Huibin Liu

Wei Chen[1]

e-mail: weichen@northwestern.edu

Department of Mechanical Engineering, Northwestern University, Evanston, IL 60208-3111

Michael Kokkolaras

Panos Y. Papalambros

Department of Mechanical Engineering, University of Michigan, Ann Arbor, MI 48109-2125

Harrison M. Kim

Department of Industrial and Enterprise Systems Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801

e-mail: hmkim@uiuc.edu

Probabilistic Analytical Target Cascading: A Moment Matching Formulation for Multilevel Optimization Under Uncertainty

Analytical target cascading (ATC) is a methodology for hierarchical multilevel system design optimization. In previous work, the deterministic ATC formulation was extended to account for random variables represented by expected values to be matched among subproblems and thus ensure design consistency. In this work, the probabilistic formulation is augmented to allow the introduction and matching of additional probabilistic characteristics. A particular probabilistic analytical target cascading (PATC) formulation is proposed that matches the first two moments of interrelated responses and linking variables. Several implementation issues are addressed, including representation of probabilistic design targets, matching responses and linking variables under uncertainty, and coordination strategies. Analytical and simulation-based optimal design examples are used to illustrate the new formulation. The accuracy of the proposed PATC formulation is demonstrated by comparing PATC results to those obtained using a probabilistic all-in-one formulation. [DOI: 10.1115/1.2205870]

Keywords: hierarchical multilevel optimization, analytical target cascading, uncertainty, probabilistic approach, design targets and consistency, coordination strategy, moments

1 Introduction

Optimization of complex systems typically involves a large number of design variables and coupled multidisciplinary analyses. The so-called all-in-one (AIO) approach, in which a large-scale optimization problem is formulated and solved with fully integrated multidisciplinary analyses (MDA), may not be practical as the MDA can be computationally expensive at each optimization iteration. It may be desirable to decompose the system into a number of subsystems each represented by an optimization subproblem. As illustrated in Fig. 1, a system decomposition can be hierarchical (Fig. 1(*a*)) or nonhierarchical (Fig. 1(*b*)).

Multidisciplinary design optimization (MDO) methodologies have been developed to support decomposed, distributed optimization in an effort to maintain disciplinary autonomy under a decentralized, multidisciplinary design environment [1,2]. Existing MDO techniques [3–7] were typically developed for nonhierarchically decomposed systems. Subsystems are optimized concurrently, while a system-level coordinator is used to take into account subsystem interactions.

Analytical target cascading (ATC) is a methodology developed for hierarchical multilevel system optimization [8–13]. ATC is intended primarily for hierarchies decomposed by objects or physical subsystems rather than by aspects or disciplines [14], as it is common in MDO. Each block in the hierarchical structure of Fig. 1(*a*) is referred to as an *element* or a *subproblem*, which can have only one *parent* element, but multiple *children* elements. The original problem is decomposed at multiple levels, while interactions among subsystems with the same parent element are considered and coordinated at the level above. ATC operates by specifying system design targets at the top level and formulating and solving a minimum deviation optimization problem[2] for each element in the hierarchy. The system design targets are often determined by means of enterprise-level decision-making models [15,16]. The process of cascading system targets to design specifications for subsystems at lower levels of the hierarchy matches the current way of meeting design targets within a corporate hierarchical organizational structure.

MDO formulations, including ATC, were originally developed for deterministic design problems. Incorporating uncertainty in a MDO formulation is complicated due to the interconnections among multiple elements that exchange information. Efforts to extend MDO to account for uncertainty have been based on integrating either robust design principles [17–22] or reliability-based techniques [23,24] into MDO formulations. However, most of the research mentioned above is developed for nonhierarchical system optimization problems, which are formulated as single- or bi-level problems.

Kokkolaras et al. [25] extended ATC to a probabilistic formulation using expected values to represent random variables communicated among elements. An efficient and accurate uncertainty propagation method was proposed, and although both means and variances of random variables were estimated for use when solving the probabilistic design subproblems, only the mean values of interconnected subproblem responses and linking variables were matched. However, matching only the mean values of random variables may be insufficient to ensure design consistency[3] under uncertainty.

In this article, we present a more general probabilistic ATC (PATC) formulation that can accommodate various representations of uncertainty in the multilevel problem. Several issues re-

[1]Corresponding author.

Contributed by the Design Automation Committee of ASME for publication in the Journal of Mechanical Design. Manuscript received June 29, 2005; final manuscript received October 26, 2005. Review conducted by Shapour Azarm. Paper presented at the ASME 2005 Design Engineering Technical Conferences and Computers and Information in Engineering Conference (DETC2005), September 24, 2005–September 28, 2005, Long Beach, California, USA.

[2]A minimum deviation optimization problem is an optimization problem that strives to minimize deviations of actual system responses from assigned target values.

[3]Design consistency means that the values of coupling responses and linking variables are matched among elements.

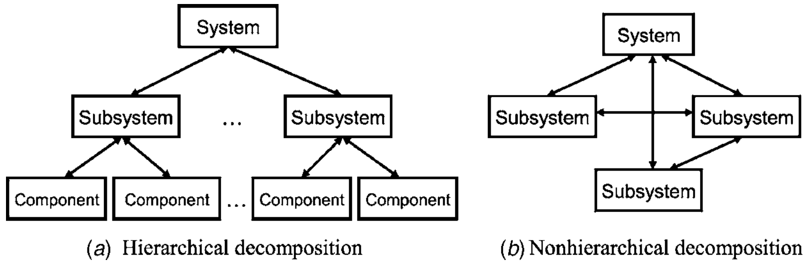


Fig. 1 System decomposition approaches

lated to the implementation of the proposed PATC formulation are examined. First, the meaning and representation of design targets under uncertainty are addressed. In our implementation, we adopt the quality engineering principle [26] to set the targets for probabilistic characteristics of engineering attributes throughout the hierarchy. Second, matching probabilistic behaviors from interrelated elements is addressed. The degree of matching probabilistic characteristics can have a large impact on efficiency of the process and accuracy of the computed design. In our implementation, design consistency is achieved by matching the first two moments of interrelated responses and linking variables. The accuracy of the proposed PATC formulation is demonstrated in case studies that compare the results of a probabilistic all-in-one (PAIO) formulation with those from PATC. Finally, we investigate empirically the potential impact of the coordination strategy on the convergence of PATC by comparing solutions and efficiency of both top-down and bottom-up strategies.

The organization of the article is as follows. In Sec. 2 we briefly review the deterministic ATC formulation and present a generalized PATC formulation. In Sec. 3 we take a close look at issues, such as uncertainty representation, design consistency, and coordination strategies. In Sec. 4 analytical and simulation-based examples are used to demonstrate the formulation. Conclusions and suggestions for future work are presented in Sec. 5.

2 Probabilistic Analytical Target Cascading Formulation

2.1 Review of the Deterministic Formulation. Product development includes a process of meeting targets $\mathbf{T}$ set by the enterprise level decision-making models [27], expressed as a deterministic all-in-one (AIO) optimization problem

$$\begin{aligned} &\text{Given} \quad \mathbf{T} \\ &\text{find} \quad \mathbf{x} \\ &\text{to minimize} \quad \|\mathbf{T} - \mathbf{r}(\mathbf{x})\| \\ &\text{subject to} \quad \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \end{aligned} \tag{1}$$

The vector $\mathbf{x}$ includes all design variables, while the vector $\mathbf{r}$ represents the system's responses. The vector $\mathbf{T}$ includes the target values for $\mathbf{r}$, fixed during the optimization process. The design objective is to find a feasible design $\mathbf{x}$ that brings the responses $\mathbf{r}$ as close as possible to the assigned targets $\mathbf{T}$. The quality of a design is measured by the deviation between $\mathbf{r}$ and $\mathbf{T}$, using some (possibly weighted) norm. In this article, we use the l_2-norm to measure deviations, but square the norms in the computational implementation of the process to avoid derivative discontinuities.

Using the concept of ATC, the AIO problem in Eq. (1) is decomposed hierarchically into elements at multiple levels. Coupling among elements is captured by linking variables. Linking variables can be design variables shared among elements with the same parent or responses from "sibling" elements at the same level [28]. Each element is a subproblem of a smaller size. Inputs to an element include its local design variables, responses from its children elements, linking variables among its children elements, and linking variables from sibling elements. The design and analysis models at multiple levels are hierarchical by nature as the output of a lower-level model becomes the input of a higher-level model.

The deterministic ATC optimization of element j at level $i(O_{ij})$ with n_{ij} children is formulated in Eq. (2). The vector $\mathbf{r}_{ij}$ represents the element's responses. The optimization variables include local design variables $\mathbf{x}_{ij}$, linking variables $\mathbf{y}_{ij}$, targets for children responses $\mathbf{r}_{(i+1)k}$, $k=1,\ldots,n_{ij}$, targets for children linking variables $\mathbf{y}_{(i+1)j}$, and tolerance optimization variables ε^r and ε^y to coordinate children responses and linking variables for design consistency. The collective optimization variables will from now on be referred to as *decision variables*. Note that element O_{ij} collects all linking variables of its children in a single vector $\mathbf{y}_{(i+1)j}$. The kth child of O_{ij} uses a selection matrix $\mathbf{S}_{(i+1)k}$, to identify which components of $\mathbf{y}_{(i+1)j}$ correspond to the linking variables $\mathbf{y}_{(i+1)k}$ of that child [29]. Similarly, the O_{ij} itself uses its selection matrix $\mathbf{S}_{ij}$ to identify the target values for its linking variables from the vector $\mathbf{y}^U_{iq}$, where q denotes its parent.

$$\begin{aligned} &\text{Given} \quad \mathbf{r}^U_{ij},\mathbf{y}^U_{iq},\mathbf{r}^L_{(i+1)k},\mathbf{y}^L_{(i+1)k},\mathbf{S}_{ij},\mathbf{S}_{(i+1)k}, k=1,\ldots,n_{ij} \\ &\text{find} \quad \mathbf{r}_{(i+1)k},\mathbf{x}_{ij},\mathbf{y}_{ij},\mathbf{y}_{(i+1)j},\varepsilon^r_{ij},\varepsilon^y_{ij},\ k=1,\ldots,n_{ij} \\ &\text{to minimize} \quad \|\mathbf{r}_{ij}-\mathbf{r}^U_{ij}\| + \|\mathbf{y}_{ij}-\mathbf{S}_{ij}\mathbf{y}^U_{iq}\| + \varepsilon^r_{ij} + \varepsilon^y_{ij} \\ &\text{subject to} \quad \sum_{k=1}^{n_{ij}} \|\mathbf{r}_{(i+1)k}-\mathbf{r}^L_{(i+1)k}\| \leq \varepsilon^r_{ij} \\ &\qquad \sum_{k=1}^{n_{ij}} \|\mathbf{S}_{(i+1)k}\mathbf{y}_{(i+1)j}-\mathbf{y}^L_{(i+1)k}\| \leq \varepsilon^y_{ij} \\ &\qquad \mathbf{g}_{ij}(\mathbf{r}_{ij},\mathbf{x}_{ij},\mathbf{y}_{ij}) \leq 0 \\ &\text{where} \quad \mathbf{r}_{ij} = \mathbf{f}_{ij}(\mathbf{r}_{(i+1)1},\ldots,\mathbf{r}_{(i+1)n_{ij}},\mathbf{x}_{ij},\mathbf{y}_{ij}) \end{aligned} \tag{2}$$

In Eq. (2), superscripts U indicate targets assigned by the parent element, while superscripts L indicate values passed from children elements. The targets for responses and linking variables of element O_{ij} are $\mathbf{r}^U_{ij}$ and $\mathbf{S}_{ij}\mathbf{y}^U_{iq}$, respectively. The actual achievable values, $\mathbf{r}^L_{(i+1)k}$ and $\mathbf{y}^L_{(i+1)k}$, are passed up to O_{ij} from its children. Solving the problem in Eq. (2), element O_{ij} finds the achievable values of its responses and linking variables that are the closest to $\mathbf{r}^U_{ij}$ and $\mathbf{S}_{ij}\mathbf{y}^U_{iq}$, respectively. O_{ij} then passes them back to its parent element as $\mathbf{r}^L_{ij}$ and $\mathbf{y}^L_{ij}$, respectively. It also determines the optimal values for its children responses and linking variables with the least inconsistency from $\mathbf{r}^L_{(i+1)k}$and $\mathbf{y}^L_{(i+1)k}$. These optimal values are passed down as targets, $\mathbf{r}^U_{(i+1)k}$ and $\mathbf{y}^U_{(i+1)j}$.

2.2 Generalized Probabilistic ATC Formulation. In a probabilistic design optimization formulation, uncertain quantities are random variables that can be characterized by a probability density function (PDF), a cumulative distribution function (CDF), or descriptors such as moments [30]. We use the superscript ν to denote probabilistic characteristics of a random variable. For example, for a normally distributed random variable $X, \mathbf{X}^\nu = [\mu_X, \sigma_X]$. Still taking the objective as meeting design targets, the probabilistic AIO (PAIO) optimization formulation is

$$\begin{aligned}
&\text{Given} \quad \mathbf{T}^\nu \\
&\text{find} \quad \mathbf{X}^\nu \\
&\text{to minimize} \quad \|\mathbf{T}^\nu - \mathbf{R}^\nu\| \\
&\text{subject to} \quad \Pr[g_m(\mathbf{X}) \leq 0] \geq \alpha_m, \; m = 1, \ldots, M \\
&\text{with} \quad \mathbf{R} = \mathbf{f}(\mathbf{X})
\end{aligned} \tag{3}$$

where M is the number of constraints. In Eq. (3), capital letters **R** and **X** are used to represent the random variables (instead of **r** and **x** used in Eq. (1)). We assume that an appropriate uncertainty propagation technique for computing $\mathbf{R}^\nu$ is available. Design constraints are posed using the probabilistic feasibility formulation [31], with α_m denoting the required reliability levels. Note that the system design targets vector $\mathbf{T}^\nu$ in Eq. (3) has a different meaning from **T**, the targets for deterministic responses in Eq. (1). In the presence of uncertainty, $\mathbf{T}^\nu$ consists of target values that correspond to the probabilistic characteristics $\mathbf{R}^\nu$. Setting the targets for probabilistic characteristics is important because variations of system performance can lead to customer dissatisfaction and additional costs to the producer. On the other hand, reducing performance variations often causes increase in the cost of product development. For example, in considering vehicle engine noise under different operating temperatures, design targets should be set for both the nominal value of engine noise and its standard deviation.

Kokkolaras et al. [25] proposed a PATC formulation, in which expected values (means) are used to represent random variables. For example, in their formulation the characteristic R^ν of a random response R is a single scalar (the expected value $E(R)$). Accordingly, design targets were only defined for the nominal values of design performance.

In this article, we provide a more general PATC formulation where any interrelated random variables (responses and linking variables) are described by general probabilistic characteristics. The formulation for element j optimization at level $i(O_{ij})$ with n_{ij} children is shown in Eq. (4) (using comma with additional subscript index to denote vector components, e.g., for the constraints).

$$\begin{aligned}
&\text{Given} \quad \mathbf{R}_{ij}^{\nu,U}, \mathbf{Y}_{iq}^{\nu,U}, \mathbf{R}_{(i+1)k}^{\nu,L}, \mathbf{Y}_{(i+1)k}^{\nu,L}, \mathbf{S}_{ij}, \mathbf{S}_{(i+1)k}, \; k = 1, \ldots, n_{ij} \\
&\text{find} \quad \mathbf{R}_{(i+1)k}^{\nu}, \mathbf{X}_{ij}^{\nu}, \mathbf{Y}_{ij}^{\nu}, \mathbf{Y}_{(i+1)j}^{\nu}, \varepsilon_{ij}^{R}, \varepsilon_{ij}^{Y}, k = 1, \ldots, n_{ij} \\
&\text{to minimize} \quad \|\mathbf{R}_{ij}^{\nu} - \mathbf{R}_{ij}^{\nu,U}\| + \|\mathbf{Y}_{ij}^{\nu} - \mathbf{S}_{ij}\mathbf{Y}_{iq}^{\nu,U}\| + \varepsilon_{ij}^{R} + \varepsilon_{ij}^{Y} \\
&\text{subject to} \quad \sum_{k=1}^{n_{ij}} \|\mathbf{R}_{(i+1)k}^{\nu} - \mathbf{R}_{(i+1)k}^{\nu,L}\| \leq \varepsilon_{ij}^{R} \\
&\qquad \sum_{k=1}^{n_{ij}} \|\mathbf{S}_{(i+1)k}\mathbf{Y}_{(i+1)j}^{\nu} - \mathbf{Y}_{(i+1)k}^{\nu,L}\| \leq \varepsilon_{ij}^{Y} \\
&\qquad \Pr[g_{ij,m}(\mathbf{R}_{ij}, \mathbf{X}_{ij}, \mathbf{Y}_{ij}) \leq 0] \geq \alpha_{ij,m}, \; m = 1, \ldots, M \\
&\text{where} \quad \mathbf{R}_{ij} = \mathbf{f}_{ij}(\mathbf{R}_{(i+1)1}, \ldots, \mathbf{R}_{(i+1)n_{ij}}, \mathbf{X}_{ij}, \mathbf{Y}_{ij})
\end{aligned} \tag{4}$$

The above formulation is generally applicable to all the elements of the multilevel hierarchy. Nevertheless, top- and bottom-level problems in PATC are special cases of this formulation. At the top level of the hierarchy ($i=0$), there is only one element O_0 (the element index is thus dropped at this level) and there are no linking variables; also, the systems' design targets $\mathbf{R}_0^{\nu,U}$ are defined in the vector $\mathbf{T}^\nu$ in Eq. (3). Elements at the bottom level do not have any children; thus, the first two constraints in Eq. (4) and the ε-variables are eliminated. The structure of PATC is very similar to the deterministic one in Eq. (2), except that the targets on responses and linking variables in the objective are now expanded to include probabilistic characteristics of these quantities, and the consistency constraints are augmented to match individual probabilistic characteristics of children responses and linking variables.

3 PATC Implementation Issues

Three major issues need to be addressed to ensure effective and efficient implementation of the PATC formulation. The first question is what probabilistic characteristics should be used to represent the system-level responses and how to assign associated target values. The second issue relates to the choice of probabilistic characteristics to match all interrelated random responses and linking variables for ensuring design consistency under uncertainty. These two issues are discussed in Sec. 3.1 as they are both related to the choice of probabilistic characteristics. This discussion leads to the particular PATC formulation that matches the first two moments of element responses and linking variables, presented in Sec. 3.2. The third issue, addressed in Sec. 3.3, relates to choices of coordination strategies for the PATC process, considering that initial uncertainty information (to be propagated throughout the multilevel hierarchy during the iterative ATC process) may be available at different levels.

3.1 Choice of Probabilistic Characteristics. Random variable representation in a PATC formulation depends on the choice of probabilistic characteristics. Moments (e.g., mean and variance) are popular and efficient descriptors of random variables. To set up targets for probabilistic characteristics, quality engineering principles can be adopted to meet robustness and reliability targets on performance at various levels. The robust design principle is accomplished by bringing the performance mean to its target while reducing the performance variance [31–33]. Following the robust design principle, targets can be set for the mean and standard deviation of design responses, denoted as $\mathbf{T}^\mu$ and $\mathbf{T}^\sigma$ for $\mu_{\mathbf{R}}$ and $\sigma_{\mathbf{R}}$ correspondingly, at the top system level. When considering design reliability, targets can be set either for a reliability level α or for a percentile performance [34] corresponding to α.

Determining the target values for system-level probabilistic characteristics will require introducing an enterprise-driven design approach [15,16,35,36], which is not the focus of this study. The enterprise decision making model captures the impact of quality engineering characteristics (mean, robustness, reliability, etc.) on product demand and cost, and sets up the targets based on the tradeoffs.

In PATC, targets set at the top level are cascaded to lower levels to guide quality engineering practice throughout the hierarchy. In particular, if targets for mean and standard deviation are set for system level performance, cascading targets on mean and standard deviation throughout the multilevel hierarchy guides robust design efforts at all levels.

In addition to matching assigned targets from a higher level, it is critical to also match the interrelated probabilistic characteristics (responses and linking variables) for ensuring design consistency under uncertainty. Matching the whole distribution is impractical as the computational cost of the optimization subproblems would increase substantially. For distributions with negligible higher-order moments, matching only the first two moments should be sufficient. Otherwise, the former may need to be included. In most situations, matching the first four moments would be sufficient but not affordable as the approximation of higher order moments requires additional computational effort or

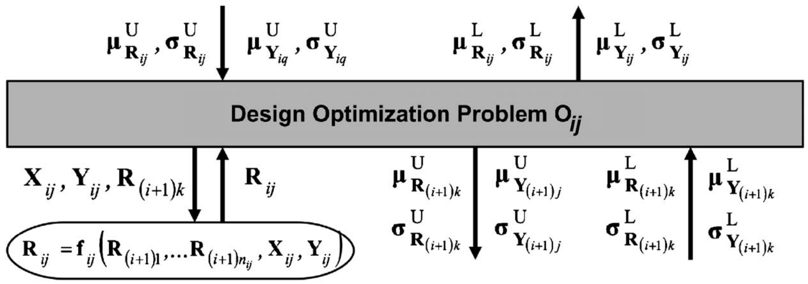


Fig. 2 Information flow for particular PATC formulation

larger number of samples. Prior knowledge or educated guess of the distribution type is useful for selecting appropriate characteristics to match.

3.2 PATC Formulation Based on Matching Mean and Variance.

The particular implementation of the general PATC formulation presented in this article sets targets on mean and standard deviation for element performance based on robust design considerations and matches the first two moments of interrelated responses and linking variables. The information flow of the design optimization problem for element j at level i (element O_{ij}) is shown in Fig. 2.

$\mathbf{R}_{ij}$ and $\mathbf{Y}_{ij}$ are vectors of random responses and linking variables, respectively. $\mathbf{R}_{ij}$ are evaluated using analysis or simulation models $\mathbf{R}_{ij}=\mathbf{f}_{ij}(\mathbf{R}_{(i+1)1},\ldots,\mathbf{R}_{(i+1)n_{ij}},\mathbf{X}_{ij},\mathbf{Y}_{ij})$. Targets for mean and standard deviation of $\mathbf{R}_{ij}$ and $\mathbf{Y}_{ij}$ are assigned by the parent element as $[\boldsymbol{\mu}^U_{R_{ij}},\boldsymbol{\sigma}^U_{R_{ij}}]$and $[\boldsymbol{\mu}^U_{Y_{iq}},\boldsymbol{\sigma}^U_{Y_{iq}}]$, respectively. Achievable values of mean and standard deviation for $\mathbf{R}_{ij}$ and $\mathbf{Y}_{ij}$ are the outputs of the optimization problem for element O_{ij}, feeding back to its parent element as $[\boldsymbol{\mu}^L_{R_{ij}},\boldsymbol{\sigma}^L_{R_{ij}}]$ and $[\boldsymbol{\mu}^L_{Y_{ij}},\boldsymbol{\sigma}^L_{Y_{ij}}]$. Similarly, achievable values of its children element responses and linking variables are passed to O_{ij} as $[\boldsymbol{\mu}^L_{R_{(i+1)k}},\boldsymbol{\sigma}^L_{R_{(i+1)k}}]$ and $[\boldsymbol{\mu}^L_{Y_{(i+1)k}},\boldsymbol{\sigma}^L_{Y_{(i+1)k}}]$, and must be taken into account for consistency. The optimization problem for element O_{ij} is solved to find the optimum values of the probabilistic characteristics (not limited to the first two moments) of its local design variables $\mathbf{X}_{ij}$ and to determine the target values for the responses and linking variables $[\boldsymbol{\mu}^U_{R_{(i+1)k}},\boldsymbol{\sigma}^U_{R_{(i+1)k}}]$ and $[\boldsymbol{\mu}^U_{Y_{(i+1)j}},\boldsymbol{\sigma}^U_{Y_{(i+1)j}}]$, respectively, of its children elements.

$$
\begin{aligned}
\text{Given}\quad & \boldsymbol{\mu}^U_{\mathbf{R}_{ij}},\boldsymbol{\sigma}^U_{\mathbf{R}_{ij}},\boldsymbol{\mu}^U_{\mathbf{Y}_{iq}},\boldsymbol{\sigma}^U_{\mathbf{Y}_{iq}},\boldsymbol{\mu}^L_{\mathbf{R}_{(i+1)k}},\boldsymbol{\sigma}^L_{\mathbf{R}_{(i+1)k}},\boldsymbol{\mu}^L_{\mathbf{Y}_{(i+1)k}},\\
& \boldsymbol{\sigma}^L_{\mathbf{Y}_{(i+1)k}},\mathbf{S}_{ij},\mathbf{S}_{(i+1)k},\ k=1,\ldots,n_{ij}\\
\text{find}\quad & \boldsymbol{\mu}_{\mathbf{R}_{(i+1)k}},\boldsymbol{\sigma}_{\mathbf{R}_{(i+1)k}},\mathbf{X}^{\nu}_{ij},\boldsymbol{\mu}_{\mathbf{Y}_{ij}},\boldsymbol{\sigma}_{\mathbf{Y}_{ij}},\boldsymbol{\mu}_{\mathbf{Y}_{(i+1)j}},\boldsymbol{\sigma}_{\mathbf{Y}_{(i+1)j}},\quad \varepsilon^{\boldsymbol{\mu}_\mathbf{R}}_{ij},\varepsilon^{\boldsymbol{\sigma}_\mathbf{R}}_{ij},\\
& \varepsilon^{\boldsymbol{\mu}_\mathbf{Y}}_{ij},\varepsilon^{\boldsymbol{\sigma}_\mathbf{Y}}_{ij},\ k=1,\ldots,n_{ij}\\
\text{to minimize}\quad & \|\boldsymbol{\mu}_{\mathbf{R}_{ij}}-\boldsymbol{\mu}^U_{\mathbf{R}_{ij}}\|+\|\boldsymbol{\sigma}_{\mathbf{R}_{ij}}-\boldsymbol{\sigma}^U_{\mathbf{R}_{ij}}\|+\|\boldsymbol{\mu}_{\mathbf{Y}_{ij}}-\mathbf{S}_{ij}\boldsymbol{\mu}^U_{\mathbf{Y}_{iq}}\|\\
& +\|\boldsymbol{\sigma}_{\mathbf{Y}_{ij}}-\mathbf{S}_{ij}\boldsymbol{\sigma}^U_{\mathbf{Y}_{iq}}\|+\varepsilon^{\boldsymbol{\mu}_\mathbf{R}}_{ij}+\varepsilon^{\boldsymbol{\sigma}_\mathbf{R}}_{ij}+\varepsilon^{\boldsymbol{\mu}_\mathbf{Y}}_{ij}+\varepsilon^{\boldsymbol{\sigma}_\mathbf{Y}}_{ij}\\
\text{subject to}\quad & \sum_{k=1}^{n_{ij}}\|\boldsymbol{\mu}_{\mathbf{R}_{(i+1)k}}-\boldsymbol{\mu}^L_{\mathbf{R}_{(i+1)k}}\|\leqslant\varepsilon^{\boldsymbol{\mu}_\mathbf{R}}_{ij}\\
& \sum_{k=1}^{n_{ij}}\|\boldsymbol{\sigma}_{\mathbf{R}_{(i+1)k}}-\boldsymbol{\sigma}^L_{\mathbf{R}_{(i+1)k}}\|\leqslant\varepsilon^{\boldsymbol{\sigma}_\mathbf{R}}_{ij}\\
& \sum_{k=1}^{n_{ij}}\|\mathbf{S}_{(i+1)k}\boldsymbol{\mu}_{\mathbf{Y}_{(i+1)j}}-\boldsymbol{\mu}^L_{\mathbf{Y}_{(i+1)k}}\|\leqslant\varepsilon^{\boldsymbol{\mu}_\mathbf{Y}}_{ij}\\
& \sum_{k=1}^{n_{ij}}\|\mathbf{S}_{(i+1)k}\boldsymbol{\sigma}_{\mathbf{Y}_{(i+1)j}}-\boldsymbol{\sigma}^L_{\mathbf{Y}_{(i+1)k}}\|\leqslant\varepsilon^{\boldsymbol{\sigma}_\mathbf{Y}}_{ij}\\
& \Pr[g_{ij,m}(\mathbf{R}_{ij},\mathbf{X}_{ij},\mathbf{Y}_{ij})\leqslant 0]\geqslant\alpha_{ij,m},\quad m=1,\ldots,M\\
\text{where,}\quad & \mathbf{R}_{ij}=\mathbf{f}_{ij}(\mathbf{R}_{(i+1)1},\ldots,\mathbf{R}_{(i+1)n_{ij}},\mathbf{X}_{ij},\mathbf{Y}_{ij})
\end{aligned}
\tag{5}
$$

We emphasize that Eq. (5) is a particular PATC formulation. Even though targets and interrelated random variables are matched using the first two moments, the probabilistic characteristics of local random variables $\mathbf{X}_{ij}$ are not restricted to the first two moments. It should also be noted that in the above formulation the number of optimization variables is approximately twice as large relative to that of the formulation in Kokkolaras et al. [25] since each random variable is represented by more than one descriptor.

3.3 Coordination Strategies.

Similar to deterministic ATC, PATC is an iterative process of solving optimization subproblems until deviations of probabilistic system responses from targets cannot be reduced any further without violating system consistency. This iterative process requires an appropriate coordination strategy to ensure convergence. Michelena et al. [37] proved the convergence properties of the deterministic ATC formulation for a specific class of coordination strategies under standard convexity and smoothness assumptions. In the work of Haftka et al. [38], much milder conditions for convergence are presented for a quasi-separable structure of multidisciplinary optimization problems. The applicability of these conditions to ATC is subject of future investigations.

When dealing with uncertainties that propagate throughout the multilevel hierarchy, one question is at which level the PATC process should begin. From an organization's viewpoint, the design process should start from the highest level, as usually overall targets are assigned and cascaded down from top to bottom. On the other hand, it may be beneficial to start at the level where uncertainty cannot be reduced, i.e., at the level where we cannot control the variation of random inputs. Typically, this occurs at the bottom level, where most random design variables have known distributions. The bottom-up coordination strategy imitates the uncertainty propagation process. In this study, both strategies are tested to investigate whether the starting level has an impact on convergence speed and solution accuracy. Note that we have not conducted a theoretical study of convergence properties.

4 Examples

The first example is the geometric programming problem adopted from Kim et al. [8]. The second example is the V6 gasoline engine design problem considered in Kokkolaras et al. [25]. The major objective of the case studies is to investigate the accuracy of the proposed particular PATC formulation by comparing results obtained using it to those obtained using the PAIO formulation. It is expected that the particular PATC should yield the same optimal solution as the fully integrated methods when the

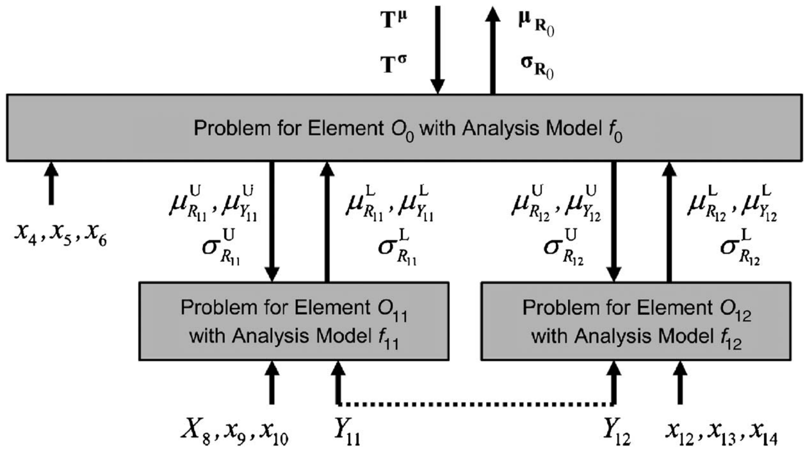


Fig. 3 Information flow in the bi-level hierarchical decomposition of geometric programming problem

first two moments can sufficiently represent the impact of uncertainties throughout a hierarchy. A preliminary investigation on coordination strategies, top-down and bottom-up, is also conducted.

4.1 Geometric Programming Problem.

4.1.1 Problem Formulation. Geometric programming problems with polynomials (polynomials with positive constants) are known to have a unique global optimum [39]. The deterministic AIO and ATC formulations of the geometric programming problem are provided in Kim et al. [8]. The PAIO problem is formulated in Eq. (6), and the purpose of solving it is to verify whether the PATC is capable of reaching the same optimal solution. In Eq. (6), capital letters are used to represent random variables, while lower cases are kept for deterministic quantities or realizations of random variables.

$$
\begin{aligned}
&\text{Given} \quad T^{\mu_{X_1}}, T^{\sigma_{X_1}}, T^{\mu_{X_2}}, T^{\sigma_{X_2}}, \sigma_{X_8}, \sigma_{X_{11}} \\
&\text{find} \quad x_4, x_5, x_7, \mu_{X_8}, x_9, x_{10}, \mu_{X_{11}}, x_{12}, x_{13}, x_{14} \geqslant 0 \\
&\text{to minimize} \quad (T^{\mu_{X_1}} - \mu_{X_1})^2 + (T^{\sigma_{X_1}} - \sigma_{X_1})^2 + (T^{\mu_{X_2}} - \mu_{X_2})^2 \\
&\qquad + (T^{\sigma_{X_2}} - \sigma_{X_2})^2 \\
&\text{subject to} \quad \Pr(g_i \leqslant 0) \geqslant \alpha_i, i = 1, \ldots, 6 \\
&\text{where } X_1 = (X_3^2 + x_4^{-2} + x_5^2)^{1/2} \; X_2 = (x_5^2 + X_6^2 + x_7^2)^{1/2} \\
&\quad X_3 = (X_8^2 + x_9^{-2} + x_{10}^{-2} + X_{11}^2)^{1/2} \; X_6 = (X_{11}^2 + x_{12}^2 + x_{13}^2 + x_{14}^2)^{1/2} \\
&\quad g_1 = \frac{X_3^{-2} + x_4^2}{x_5^2} - 1, \quad g_2 = \frac{x_5^2 + X_6^{-2}}{x_7^2} - 1, \quad g_3 = \frac{X_8^2 + x_9^{-2}}{X_{11}^2} - 1 \\
&\quad g_4 = \frac{X_8^{-2} + x_{10}^2}{X_{11}^2} - 1, \quad g_5 = \frac{X_{11}^2 + x_{12}^{-2}}{x_{13}^2} - 1, \quad g_6 = \frac{X_{11}^2 + x_{12}^2}{x_{14}^2} - 1
\end{aligned} \tag{6}
$$

In this example, the target values for the mean and the standard deviation of the system response $[X_1, X_2]$ were $[T^{\mu_{X1}}, T^{\mu_{X2}}] = [0,0]$ and $[T^{\sigma_{X1}}, T^{\sigma_{X2}}] = [0,0]$. These overall system targets are denoted as $[\mathbf{T}^{\mu}, \mathbf{T}^{\sigma}]$ in the later part of this subsection. We assume that design variables X_8 and X_{11} are independent and normally distributed with constant standard deviations $\sigma_{X_8} = \sigma_{X_{11}} = 0.1$. The required reliability level α is 99.865% for all probabilistic constraints.

The functional dependencies in Eq. (6) are used to decompose the problem into a bi-level hierarchy with two elements at the bottom level. The associated information flow is shown in Fig. 3. The randomness in X_8 and X_{11} results in uncertainties in all computed responses, each described by its mean and standard deviation. Setting $R_{0,1} = X_1, R_{0,2} = X_2, \mathbf{x}_0 = [x_4, x_5, x_7], R_{11} = X_3$, and $R_{12} = X_6$, the top-level ($i=0$) responses are computed by

$$
\mathbf{R}_0 = \begin{bmatrix} R_{0,1} \\ R_{0,2} \end{bmatrix} = \mathbf{f}_0(R_{11}, R_{12}, \mathbf{x}_0) = \begin{bmatrix} (R_{11}^2 + x_4^{-2} + x_5^2)^{1/2} \\ (x_5^2 + R_{12}^2 + x_7^2)^{1/2} \end{bmatrix} \tag{7}
$$

where a comma and additional index denote vector component.

The vectors of local design variables of two bottom-level elements (O_{11} and O_{12}) are set as $[X_8, x_9, x_{10}]$ and $[x_{12}, x_{13}, x_{14}]$, respectively. Since elements O_{11} and O_{12} share the random design variable X_{11}, it becomes a random linking variable, i.e., $Y_{11} = X_{11}$ and $Y_{12} = X_{11}$. The two bottom-level ($i=1$) element responses are computed by

$$
R_{11} = f_{11}(X_8, x_9, x_{10}, Y_{11}) = (X_8^2 + x_9^{-2} + x_{10}^{-2} + X_{11}^2)^{1/2} \tag{8}
$$

and

$$
R_{12} = f_{12}(x_{12}, x_{13}, x_{14}, Y_{12}) = (X_{11}^2 + x_{12}^2 + x_{13}^2 + x_{14}^2)^{1/2} \tag{9}
$$

The primary goal of this example is to test the effectiveness of the proposed particular PATC method. We use Monte Carlo simulation (MCS) to evaluate the first two moments of responses to avoid the influence caused by approximation methods. All probabilistic constraints are evaluated by the moment-matching method:

$$
\mu_g + k\sigma_g \leqslant 0, \tag{10}
$$

where k is a constant. Corresponding to the required reliability level set at 99.865%, k is equal to 3 in this example. μ_g and σ_g are also obtained by MCS. The probabilistic optimization models for the three elements O_0, O_{11}, and O_{12} in Fig. 3 under the particular PATC formulation are formulated in Eqs. (11)–(13), respectively. Note that since the standard deviation of the random design variable X_{11} is assumed constant (i.e., cannot be controlled), it is not included as an decision variable. In general, if we cannot control the standard deviation of a random response or a linking variable, we are forced to omit the corresponding standard deviation from the particular moment-matching formulation of Eq. (5).

O_0: Given $\quad \mathbf{T}^{\mu}, \mathbf{T}^{\sigma}, \mu^L_{R_{11}}, \sigma^L_{R_{11}}, \mu^L_{R_{12}}, \sigma^L_{R_{12}}, \mu^L_{Y_{11}}, \mu^L_{Y_{12}}$

find $x_4, x_5, x_7, \mu_{R_{11}}, \sigma_{R_{11}}, \mu_{R_{12}}, \sigma_{R_{12}}, \mu_{Y_1}, \varepsilon^{\mu_R}, \varepsilon^{\sigma_R}, \varepsilon^{\mu_Y} \geqslant 0$

to minimize $\quad \|\mathbf{T}^{\mu} - \boldsymbol{\mu}_{\mathbf{R}_0}\|_2^2 + \|\mathbf{T}^{\sigma} - \boldsymbol{\sigma}_{\mathbf{R}_0}\|_2^2 + \varepsilon^{\mu_R} + \varepsilon^{\sigma_R} + \varepsilon^{\mu_Y}$

$$\text{subject to} \quad (\mu_{R_{11}}-\mu^L_{R_{11}})^2+(\mu_{R_{12}}-\mu^L_{R_{12}})^2 \leqslant \varepsilon^{\mu_R}$$

$$(\sigma_{R_{11}}-\sigma^L_{R_{11}})^2+(\sigma_{R_{12}}-\sigma^L_{R_{12}})^2 \leqslant \varepsilon^{\sigma_R}$$

$$(\mu_{Y_1}-\mu^L_{Y_{11}})^2+(\mu_{Y_1}-\mu^L_{Y_{12}})^2 \leqslant \varepsilon^{\mu_Y}$$

$$\mu_{g_1}+3\sigma_{g_1} \leqslant 0$$

$$\mu_{g_2}+3\sigma_{g_2} \leqslant 0$$

$$\text{where, } \mathbf{R}_0=\begin{bmatrix} R_{0,1} \\ R_{0,2} \end{bmatrix}=\begin{bmatrix} (R_{11}^2+x_4^{-2}+x_5^2)^{1/2} \\ (x_5^2+R_{12}^2+x_7^2)^{1/2} \end{bmatrix}$$

$$g_1=\frac{R_{11}^{-2}+x_4^2}{x_5^2}-1, \quad g_2=\frac{x_5^2+R_{12}^{-2}}{x_7^2}-1 \tag{11}$$

O_{11}:Given $\mu^U_{R_{11}},\sigma^U_{R_{11}},\mu^U_{Y_1},\sigma^U_{Y_1},\sigma_{X_8},\sigma_{Y_{11}}(=\sigma_{X_{11}})$

find $\mu_{X_8},x_9,x_{10},\mu_{Y_{11}} \geqslant 0$

$$\text{to minimize} \quad (\mu_{R_{11}}-\mu^U_{R_{11}})^2+(\sigma_{R_{11}}-\sigma^U_{R_{11}})^2+(\mu_{Y_{11}}-\mu^U_{Y_1})^2$$

$$\text{subject to} \quad \mu_{g_3}+3\sigma_{g_3} \leqslant 0$$

$$\mu_{g_4}+3\sigma_{g_4} \leqslant 0$$

$$\text{where} \quad R_{11}=(X_8^2+x_9^{-2}+x_{10}^{-2}+Y_{11}^2)^{1/2}$$

$$g_3=\frac{X_8^2+x_9^{-2}}{Y_{11}^2}-1, \quad g_4=\frac{X_8^{-2}+x_{10}^2}{Y_{11}^2}-1 \tag{12}$$

O_{12}:Given $\mu^U_{R_{12}},\sigma^U_{R_{12}},\mu^U_{Y_1},\sigma^U_{Y_1},\sigma_{Y_{12}}(=\sigma_{X_{11}})$

find $\mu_{Y_{12}},x_{12},x_{13},x_{14} \geqslant 0$

$$\text{to minimize} \quad (\mu_{R_{12}}-\mu^U_{R_{12}})^2+(\sigma_{R_{12}}-\sigma^U_{R_{12}})^2+(\mu_{Y_{12}}-\mu^U_{Y_1})^2$$

$$\text{subject to} \quad \mu_{g_5}+3\sigma_{g_5} \leqslant 0$$

$$\mu_{g_6}+3\sigma_{g_6} \leqslant 0$$

$$\text{where} \quad R_{12}=(Y_{12}^2+x_{12}^2+x_{13}^2+x_{14}^2)^{1/2}$$

$$g_5=\frac{Y_{12}^2+x_{12}^{-2}}{x_{13}^2}-1, \quad g_6=\frac{Y_{12}^2+x_{12}^2}{x_{14}^2}-1 \tag{13}$$

Both top-down and bottom-up coordination strategies were tested. Starting from the top level requires an initial guess of $[\mu^U_{Y_{11}},\mu^U_{Y_{12}}]$ and $[\sigma^U_{Y_{11}},\sigma^U_{Y_{12}}]$ when solving O_{11}, and O_{12} for the first time. Starting from the bottom level required an initial guess of $[\mu^U_{R_{11}},\mu^U_{R_{12}}]$ and $[\sigma^U_{R_{11}},\sigma^U_{R_{12}}]$ when solving O_0 for the first time. For this example, the obtained optimal solutions were identical under both coordination strategies. The completion of optimizations in all elements is considered as one PATC *cycle*.

Table 1 Comparison of optimal designs

	Initial Point	PAIO	PATC
x_4	5.0	0.7599	0.7597
x_5	5.0	0.8676	0.8659
x_7	5.0	0.9208	0.9209
μ_{X_8}	5.0	1.1984	1.2013
x_9	5.0	0.8098	0.7912
x_{10}	5.0	0.7350	0.7229
$\mu_{X_{11}}$	5.0	1.4931	1.4737
x_{12}	5.0	0.8409	0.8419
x_{13}	5.0	2.1333	2.1080
x_{14}	5.0	1.9606	1.9344

Table 2 Comparison of optimal solutions

	PAIO	PATC	Confirmed PATC Solution
$[\mu^*_{R_{0,1}},\mu^*_{R_{0,2}}]$	[3.0875, 3.5968]	[3.1019, 3.5599]	[3.1006, 3.5488]
$[\sigma^*_{R_{0,1}},\sigma^*_{R_{0,2}}]$	[0.0874, 0.0417]	[0.0862, 0.0414]	[0.0860, 0.0413]
Objective function	22.4790	22.3038	22.2168

4.1.2 PATC Results. For each MCS, 10,000 samples were used. When the maximum value of deviation terms on ϵ in O_0 was within allowable tolerance (1.0×10^{-4}), and when each element optimization converged successfully, the whole PATC process was considered to have converged to an optimal design. The optimization algorithm used is sequential quadratic programming. For the specified tolerance of consistency (1.0×10^{-4}), 136 cycles were used to reach the convergence for both top-down and bottom-up strategies. The optimal design and system-level responses obtained by starting the PATC from the top level are listed in Tables 1 and 2, respectively. Table 3 compares targets assigned by O_0 for the mean of the linking variable and the actual values obtained by O_{11} and O_{12}.

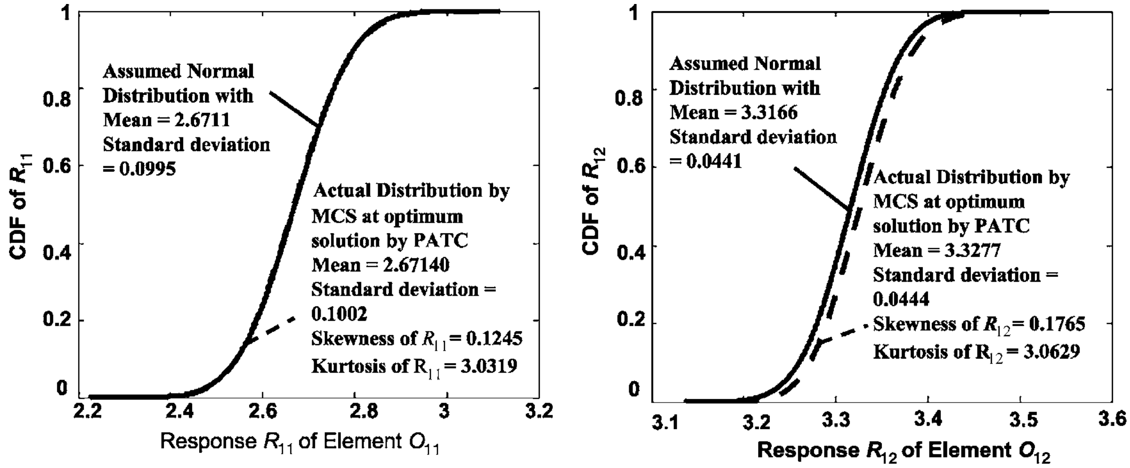
The PAIO solutions are included to verify the accuracy of the proposed PATC formulation. Tables 1 and 2 show that PATC converges to the same optimal solution as that obtained by PAIO. Table 3 shows that the optimal mean value of the shared design variable between the two coupled elements O_{11} and O_{12} is consistent.

Since only the first two moments were matched during the PATC process, the optimal solution was verified by substituting the optimal design point back into fully integrated analysis models in PAIO and computing the true values of $[\mu^*_{R_{0,1}},\mu^*_{R_{0,2}}]$ and $[\sigma^*_{R_{0,1}},\sigma^*_{R_{0,2}}]$. The results are listed in the last column in Table 2. They are sufficiently close to those from PATC, indicating that the use of the first two moments for matching probabilistic behaviors is sufficient for this example. True distributions of R_{11} and R_{12} obtained from MCS using 100,000 samples are compared to those incorporating the first two moments only (i.e., assuming normal distributions in O_0) in Fig. 4. These plots further illustrates that matching the first two moments in PATC is sufficient in this case as the higher-order (the third order and above) moments of lower-level element responses R_{11} and R_{12} are all quite small and the linking variable X_{11} is also normally distributed. To investigate how close a distribution is to a normal distribution, a normal probability plot [40] was used (not included in this article). It was found that both R_{11} and R_{12} can be considered asnormally distributed.

PATC reached the same optimum when starting from different initial points. We tested different values of weighting factors for the terms of the objective function in the problem formulation for element O_0. If weighing factors for the consistency terms (ε) are too large, e.g., 1000, the PATC formulation quickly converges to a consistent but suboptimal solution. With constant weighting factors, we observed that PATC converges to the global optimum, but many cycles are needed to fine-tune the search so as to meet the

Table 3 Comparison of linking variable mean values

	Target Value by O_0	Actual Value at O_{11}	Actual Value at O_{12}
Linking variable	1.4735	1.4834	1.4640



(*a*) Verification of distribution of R_{11}

(*b*) Verification of distribution of R_{12}

Fig. 4 Comparison of actual CDFs of responses of O_{11} and O_{12}

consistency of children elements optimization.

Although not shown explicitly in the formulations, weighting factors are introduced to capture tradeoffs among different deviation terms in the objective function and consistency constraints. Comparing to the fully integrated method, the PATC is usually more sensitive to weighting factors as relative importance needs to be determined not only among top-level responses but also among responses from lower levels. Moreover, in each element, weighting factors needs to be specified not only for different responses but also for matching linking variables, and other tolerance variables. Therefore, more efforts are needed to assigning weighting factors in the PATC. A heuristic adaptive weighting scheme that increases the values of weighting factors for the deviation terms on ϵ with the increase of PATC cycles was used in this work. A formal method for setting proper weights for element responses and linking variables in deterministic ATC can be found in Michalek et al. [29], Tosserams et al. [41], and Kim et al. [42].

4.2 Piston-Ring/Cylinder-Liner Design Problem.

4.2.1 Problem Formulation. To investigate the validity of moment-matching when element responses are known to be not normally distributed, and to investigate the performance of top-down and bottom-up coordination strategies, we use the same example as in Kokkolaras et al. [25]. The piston-ring/cylinder-liner subassembly is designed to minimize fuel consumption of a V6 engine while satisfying reliability requirements on oil consumption, blow-by, and liner wear rate. A target reliability level is chosen as 99.87%, corresponding to a reliability index as 3. The PAIO problem formulation is given in Eq. (14).

$$\text{Given}\quad T^{\mu},T^{\sigma},\sigma_{X_1},\sigma_{X_2}$$

$$\text{find}\quad \mu_{X_1},\mu_{X_2},x_3,x_4$$

$$\text{to minimize}\quad (T^{\mu}-\mu_{R_{\text{fuel}}})^2+(T^{\sigma}-\sigma_{R_{\text{fuel}}})^2$$

$$\text{subject to}\quad \Pr(R_{\text{wear}}\leqslant 2.4\times 10^{-12}\ \text{m}^3/\text{s})\geqslant 99.87\%$$

$$\Pr(R_{\text{blow by}}\leqslant 4.25\times 10^{-5}\ \text{kg/s})\geqslant 99.87\%$$

$$\Pr(R_{\text{oil}}\leqslant 15.3\ \text{g/hr})\geqslant 99.87\%$$

$$4\ \mu\text{m}\leqslant \mu_{X_1}\leqslant 7\ \mu\text{m},4\ \mu\text{m}\leqslant \mu_{X_2}\leqslant 7\ \mu\text{m}$$

$$80\ \text{GPa}\leqslant x_3\leqslant 340\ \text{GPa},\quad 150\ \text{BHV}\leqslant x_4\leqslant 240\ \text{BHV}$$

$$\text{where}\ R_{\text{fuel}}=f_{\text{fuel}}(R_{\text{power loss}}),\quad R_{\text{power loss}}=f_{\text{power loss}}(X_1,X_2,x_3,x_4),$$

$$R_{\text{wear}}=f_{\text{wear}}(X_1,X_2,x_3,x_4),R_{\text{blow by}}=f_{\text{blow by}}(X_1,X_2,x_3,x_4),$$

$$R_{\text{oil}}=f_{\text{oil}}(X_1,X_2,x_3,x_4). \tag{14}$$

The targets for the top system performance (fuel consumption) were set as $[T^{\mu},T^{\sigma}]=[0,0]$. The random design variables X_1 and X_2 denote piston-ring and cylinder-liner surface roughness, respectively. Based on measurements, they are normally distributed with $\sigma_{X_1}=\sigma_{X_2}=1\ \mu$m. The deterministic design variables x_3 and x_4 denote Young's modulus and hardness of the liner material, respectively. There are three reliability constraints related to the subassembly's performance, i.e., liner wear rate (R_{wear}), blow-by ($R_{\text{blow by}}$), and oil consumption (R_{oil}).

The problem is decomposed into a two-level hierarchy with only one element at each level. The four design variables in Eq. (14) are inputs to the bottom-level element, whose responses $\boldsymbol{R}_1$ include power loss due to ring/liner friction ($R_{\text{power loss}}$), liner wear rate (R_{wear}), blow-by ($R_{\text{blow by}}$), and oil consumption (R_{oil}). The top-level element takes only the power loss response $R_{1,1}$ ($=R_{\text{power loss}}$) as input and provides fuel consumption as the system level response $R_0(=R_{\text{fuel}})$. Once again, a comma and additional subscript index denotes a vector component.

Following the PATC formulation presented in Sec. 3.2, the mean and standard deviation are selected as the probabilistic characteristics for responses. The top level problem O_0 is formulated in Eq. (15). Because there is only one element at each level, there are no linking variables in this example. PATC constraints in O_0 are used to ensure consistency of child element responses with obtained targets.

$$O_0\text{:Given}\quad T^{\mu},T^{\sigma},\mu^{L}_{R_{1,1}},\sigma^{L}_{R_{1,1}}$$

$$\text{find}\quad \mu_{R_{1,1}},\sigma_{R_{1,1}},\varepsilon^{\mu},\varepsilon^{\sigma}$$

$$\text{to minimize}\quad (T^{\mu}-\mu_{R_0})^2+(T^{\sigma}-\sigma_{R_0})^2+\varepsilon^{\mu}+\varepsilon^{\sigma}$$

$$\text{subject to}\quad (\mu_{R_{1,1}}-\mu^{L}_{R_{1,1}})^2\leqslant\varepsilon^{\mu}$$

$$(\sigma_{R_{1,1}}-\sigma^{L}_{R_{1,1}})^2\leqslant\varepsilon^{\sigma}$$

$$\text{where}\quad R_0=R_{\text{fuel}}=f_0(R_{1,1}). \tag{15}$$

Table 4 Comparison of optimal designs (scenario 1—using non-normalized objective functions)

Design variable	Description	Initial point	PAIO	PATC (top-down)	PATC (bottom-up)
$\mu_{X_1}(\mu\text{m})$	Ring surface roughness	1.0	4.0	4.0063	4.0
$\mu_{X_2}(\mu\text{m})$	Liner surface roughness	1.0	6.1193	6.1130	6.1193
$x_3(\text{GP}_a)$	Liner Young's modulus	100	80	80.0445	80
x_4 (BHV)	Liner hardness	100	240	240	240

Table 6 Comparison of optimal designs (scenario 2—using normalized objective functions)

Design variable	Description	Initial point	PAIO	PATC (top-down)	PATC (bottom-up)
$(\mu_{X_1})(\mu\text{m})$	Ring surface roughness	1.0	7.0	7.0	7.0
$(\mu_{X_2})(\mu\text{m})$	Liner surface roughness	1.0	7.0	7.0	7.0
$x_3(\text{GP}_a)$	Liner Young's modulus	100	340	340	340
x_4(BHV)	Liner hardness	100	234.7299	234.7299	234.7299

$$O_1\text{:Given}\quad \mu_{R_{1,1}}^U,\sigma_{R_{1,1}}^U,\sigma_{X_1},\sigma_{X_2}$$

$$\text{find}\quad \mu_{X_1},\mu_{X_2},x_3,x_4$$

$$\text{to minimize}\quad (\mu_{R_{1,1}}-\mu_{R_{1,1}}^U)^2+(\sigma_{R_{1,1}}-\sigma_{R_{1,1}}^U)^2$$

$$\text{subject to}$$

$$\Pr(R_{\text{wear}} \leqslant 2.4\times 10^{-12}\ \text{m}^3/\text{s}) \geqslant 99.87\%$$

$$\Pr(R_{\text{blow by}} \leqslant 4.25\times 10^{-5}\ \text{kg/s}) \geqslant 99.87\%$$

$$\Pr(R_{\text{oil}} \leqslant 15.3\ \text{g/hr}) \geqslant 99.87\%$$

$$4\ \mu\text{m} \leqslant \mu_{X_1} \leqslant 7\ \mu\text{m}, 4\ \mu\text{m} \leqslant \mu_{X_2} \leqslant 7\ \mu\text{m}$$

$$80\ \text{GPa} \leqslant x_3 \leqslant 340\ \text{GPa},\quad 150\ \text{BHV} \leqslant x_4 \leqslant 240\ \text{BHV},$$

$$\text{where}\quad \mathbf{R}_1=\begin{bmatrix} R_{1,1}\\ R_{1,2}\\ R_{1,3}\\ R_{1,4}\end{bmatrix}=\begin{bmatrix} R_{\text{power loss}}\\ R_{\text{wear}}\\ R_{\text{blow by}}\\ R_{\text{oil}}\end{bmatrix}=\mathbf{f}_1(X_1,X_2,x_3,x_4). \tag{16}$$

The formulation of the bottom level element O_1 according to the PATC process is shown in Eq. (16). The problem for element O_1 is solved by the sequential optimization and reliability assessment (SORA) method [43]. The means and standard deviations in Eqs. (15) and (16) were evaluated by MCS. To ease the computational burden, the analysis models in O_0 and O_1 are surrogate models built using the cross-validated moving least squares method [25]. The convergence criteria for the PATC were the same as those for Example 1. For comparison, the PAIO problem was also solved using the SORA method [43].

4.2.2 PATC Results. Two probabilistic optimization scenarios were examined. In the first scenario, the mean and standard deviation values are used in the objective functions of the PAIO (Eq. (14)) and PATC problems (Eqs. (15) and (16)) without normalization. The optimal solution results using top-down and bottom-up strategies are listed in Tables 4 and 5.

As shown in Table 4, the two coordination strategies lead to the same optimal design, which is also the same as that from the PAIO formulation. In Table 5, the optimal response moments under columns "PATC" are confirmed by substituting the optimal points back into the PAIO fully integrated analysis models for R_{fuel} and $R_{\text{power loss}}$. The values of obj^* were computed as $(T^\mu-\mu_{R_{\text{fuel}}}^*)^2+(T^\sigma-\sigma_{R_{\text{fuel}}}^*)^2$. The confirmed solutions in Table 5 are very close to those from PATC and PAIO, indicating that use of the first two moments for matching probabilistic behaviors is sufficient for this example.

The variance of fuel consumption in Table 5, $\sigma_{R_{\text{fuel}}}^*$, is very small compared to the optimal mean value $\mu_{R_{\text{fuel}}}^*$. Using non-normalized objective functions in Eq. (15) is biased towards minimizing the mean of fuel consumption. The results obtained are meaningful because they are close to those from [25], which were obtained only considering the mean values of probabilistic performance. Based on 100,000 samples from MCS, the actual reliabilities of three probabilistic constraints on liner wear rate, blow by, and oil consumption are 100%, 100%, and 99.83%, respectively.

To overcome the above bias caused by different scales of the mean and standard deviation values, in the second scenario, the mean and standard deviation values in the objective functions of the PAIO and PATC problem formulations were normalized. The mean and standard deviation terms of the top-level element response (fuel consumption) in O_0 are normalized by their best achievable values, $\mu_{R_{\text{fuel}}}^{\min}=0.5359$ and $\sigma_{R_{\text{fuel}}}^{\min}=0.0033$, respectively. Similarly, the mean and standard deviation terms of the bottom-level element response (power loss) in O_1 are normalized with $\mu_{R_{\text{power loss}}}^{\min}=0.3916$ and $\sigma_{R_{\text{power loss}}}^{\min}=0.0129$. The obtained optimal design and objective values from the PATC are compared with those from the PAIO in Tables 6 and 7.

In scenario 2, top-down and bottom-up strategies also converged to identical optimal points and the confirmed optimal objective values are very close to those from the PAIO. Using the normalized objective functions results in emphasizing the standard deviation of fuel consumption, and the probabilistic optimization reaches a different optimal solution from that found in the first scenario. For this bi-level problem with only one element at each level, the top-down PATC converges after two cycles while

Table 5 Comparison of optimal objective function values (scenario 1—using non-normalized objective functions)

	PAIO	PATC (top-down)	Confirmed solution (top-down)	PATC (bottom-up)	Confirmed solution (bottom-up)
obj_j^*	2.8554×10^{-1}	2.8537×10^{-1}	2.855×10^{-1}	2.8537×10^{-1}	2.8549×10^{-1}
$\mu_{R_{\text{fuel}}}^*$	5.343×10^{-1}	5.3375×10^{-1}	5.3429×10^{-1}	5.3375×10^{-1}	5.3425×10^{-1}
$\sigma_{R_{\text{fuel}}}^*$	8.391×10^{-3}	8.6527×10^{-3}	8.3825×10^{-3}	8.6527×10^{-3}	8.3911×10^{-3}
$\mu_{R_{\text{power loss}}}^*$	3.922×10^{-1}	3.9175×10^{-1}	3.9234×10^{-1}	3.9159×10^{-1}	3.9218×10^{-1}
$\sigma_{R_{\text{power loss}}}^*$	3.448×10^{-2}	3.5163×10^{-2}	3.4438×10^{-2}	3.5204×10^{-2}	3.4478×10^{-2}

Table 7 Comparison of optimal objective function values (scenario 2—using normalized objective functions)

	PAIO	PATC (top-down)	Confirmed solution (top-down)	PATC (bottom-up)	Confirmed solution (bottom-up)
obj^*	1.9326×10^{-1}	1.9074×10	1.9369×10	1.8629×10	1.9369×10
$\mu^*_{R_{\text{fuel}}}$	5.5131×10^{-1}	5.5016×10^{-1}	5.5132×10^{-1}	5.5100×10^{-1}	5.5132×10^{-1}
$\sigma^*_{R_{\text{fuel}}}$	3.0855×10^{-3}	3.0487×10^{-3}	3.0931×10^{-3}	2.9615×10^{-3}	3.0931×10^{-3}
$\mu^*_{R_{\text{power loss}}}$	4.6008×10^{-1}	4.5555×10^{-1}	4.6011×10^{-1}	4.5902×10^{-1}	4.6011×10^{-1}
$\sigma^*_{R_{\text{power loss}}}$	1.1863×10^{-2}	1.1615×10^{-2}	1.1892×10^{-2}	1.1005×10^{-2}	1.1892×10^{-2}

the bottom-up PATC converges after one cycle.

The actual PDF of the power loss is plotted and compared to a normal PDF in Fig. 5 for both scenarios. In both cases, the actual PDF of the power loss has two modes. However, matching the first two moments in PATC seems to be sufficient for this example and can lead to the same solution as that of the PAIO. The main reason is that the top-level analysis model $f_{\text{fuel}}\ (R_{\text{power loss}})$ is nearly linear. The first two moments of fuel consumption are close to linear functions of the first two moments of power loss, regardless of the actual distribution of the power loss. Therefore, even for nonlinear models, it is sufficient to consider the first two moments as long as the first two moments of lower-level responses dominate the uncertainty propagation on higher-level responses.

Because the objective functions of the optimization problems for elements at various levels usually involve multiple deviation items, we find that special care must be taken when selecting the starting point, weighting factors, and normalization technique. Different starting points should be used if local optima are suspected. For the tolerance variables (ε^{μ} and ε^{σ}), weighting factors can neither be too large nor too small. Large weights may trap the optimum at a consistent but inferior solution after the first few cycles, while small weights may cause slow convergence to a consistent solution.

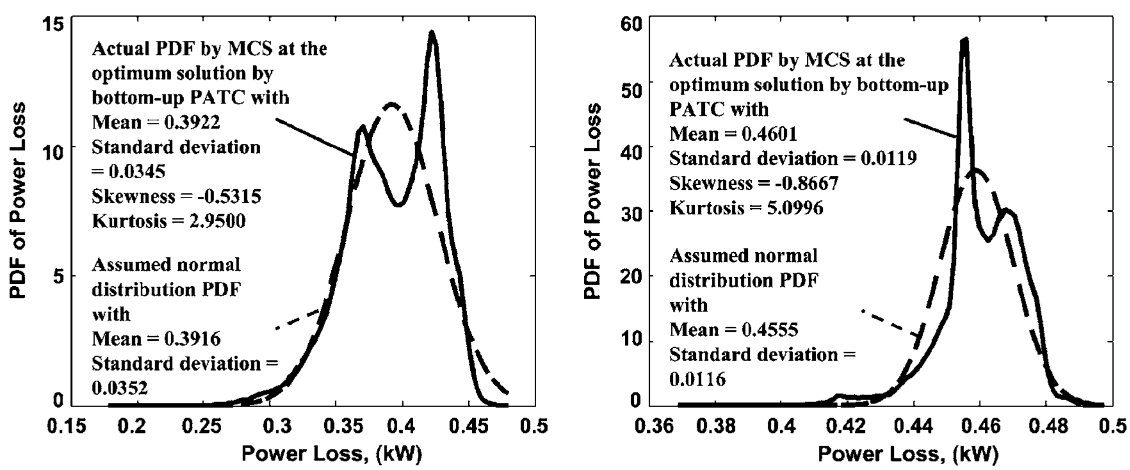
5 Conclusions

We extended previous work on ATC under uncertainty to a more general formulation. Specifically, we addressed the issue of dealing with design targets in a probabilistic framework, and, following established quality engineering principles, we proposed a particular PATC formulation that matches the first two moments of random responses and linking variables with assigned targets.

An important issue related to the accuracy of the design computed using PATC is how many moments are sufficient to match random responses and linking variables. Based on our empirical studies using two examples, when matching the first two moments of random variables, PATC converges to the same optimal solution as PAIO under two conditions: (1) When the distributions of all matching quantities are close to normal distributions (i.e., the true mean and variance of matching quantities are close to those of assumed normal distributions, as observed in the geometric programming problem), and (2) when the first two moments to be matched have dominating impact on the optimal solution, as observed in the ring/liner problem; otherwise, PATC may lead to a different optimum with an inferior objective function value. In that case, higher-order moments may need to be included and matched in the PATC formulation. Similarly, targets on correlation parameters between performance responses that are not independent with respect to the same uncertainty source should be considered. On the other hand, including additional targets increases the dimension of the parent element optimization problem.

We also need to point out that the PATC objective functions often involve multiple deviation terms. When multiple optimal solutions exist, a situation that occurs often in robust design, PATC yields the same optimal objective function value as that from PAIO, but the two approaches may yield different optimal designs. As with ATC and nonlinear optimization problems in general, local solutions may be obtained by PATC.

Future research may be conducted in the following directions. First, distributions of random variables in PATC are usually not known beforehand, and so it is desirable to create an efficient technique to determine when higher-order moments are necessary. Second, the number of decision variables in each optimization subproblem increases when higher-order moments are matched.



(*a*) PDF of $R_{\text{power loss}}$ obtained using the objective function without normalization

(*b*) PDF of $R_{\text{power loss}}$ obtained using the normalized objective function

Fig. 5 Verification of distributions of power loss in two scenarios

The impact of the number of moments used on convergence rate must be assessed. Third, convergence properties of alternative coordination strategies should be investigated. Finally, techniques can be developed to identify multiple solutions in upper-level problems so that multiple candidate targets can be used to explore the design solutions in lower-level problems.

Acknowledgment

This work was supported by NSF through a collaborative research grant between Northwestern University (DMI-0503781) and the University of Michigan (DMI-0503737). Additional support from the Ford University Research Program for Dr. Kim's postdoctoral work at Northwestern University and from the U.S. Army funded Automotive Research Center at the University of Michigan is also greatly appreciated.

References

[1] Balling, R. J., and Sobieszczanski-Sobieski, J., 1996, "Optimization of Coupled Systems: A Critical Overview of Approaches," AIAA J., **34**(1), pp. 6–17.

[2] Alexandrov, N. M., and Lewis, R. M., 1999, "Comparative Properties of Collaborative Optimization and Other Approaches to MDO," ICASE Report, No. 99-24, NASA/CR-1999-209354.

[3] Cramer, E. J., Dennis, J. E., Frank, P. D., Lewis, R. M., and Shubin, G. R., 1994, "Problem Formulation for Multidisciplinary Optimization," SIAM J. Optim., **4**(4), pp. 754–776.

[4] Kroo, I., Altus, S., Braun, R., Gage, P., and Sobieski, I., 1994, "Multidisciplinary Optimization Methods for Aircraft Preliminary Design," 5th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, paper no. AIAA 94-4325, Panama City, Florida.

[5] Braun, R. D., Kroo, I. M., and Moore, A. A., 1996, "Use of the Collaborative Optimization Architecture for Launch Vehicle Design," 6th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, paper no. AIAA 96-4018, Bellevue, WA.

[6] Kroo, I., 1997, "MDO Application in Preliminary Design: Status and Directions," AIAA Pap. 97–1408.

[7] Renaud, J. E., and Tappeta, R. V., 1997, "Multiobjective Collaborative Optimization," ASME J. Mech. Des., **119**(3), pp. 403–411.

[8] Kim, H. M., Michelena, N. F., Papalambros, P. Y., and Jiang, T., 2000, "Target Cascading in Optimal System Design," *Proceedings of the ASME Design Engineering Technical Conferences*, paper no. DETC 2000/DAC-14265, Baltimore, MD, September 10–13, 2000, 2, pp. 293–302; Also in 2003, ASME J. Mech. Des., **125**(3), pp. 474–480.

[9] Kim, H. M., 2001, Target Cascading in Optimal System Design, Ph.D. thesis, University of Michigan, Ann Arbor, Michigan.

[10] Kim, H. M., Rideout, D. G., Papalambros, P. Y., and Stein, J. L., 2001, "Analytical Target Cascading in Automotive Vehicle Design," *Proceedings of the ASME Design Engineering Technical Conferences*, paper no. DETC 2001/DAC-21079, Pittsburgh, PA, 2, pp. 661–670; Also in 2003, ASME J. Mech. Des., **125**(3), pp. 481–489.

[11] Kim, H. M., Kokkolaras, M., Louca, L. S., Delagrammatikas, G. J., Michelena, N. F., Filipi, Z. S., Papalambros, P. Y., Stein, J. L., and Assanis, D. N., 2002, "Target Cascading in Vehicle Redesign: A Class VI Truck Study," Int. J. Veh. Des., **29**(3), pp. 199–225.

[12] Kokkolaras, M., Fellini, R., Kim, H. M., Michelena, N. F., and Papalambros, P. Y., 2002, "Extension of the Target Cascading Formulation to the Design of Product Families," Struct. Multidiscip. Optim., **24**(4), pp. 293–301.

[13] Kokkolaras, M., Louca, L. S., Delagrammatikas, G. J., Michelena, N. F., Filipi, Z. S., Papalambros, P. Y., Stein, J. L., and Assanis, D. N., 2004, "Simulation-Based Optimal Design of Heavy Trucks by Model-Based Decomposition: An Extensive Analytical Target Cascading Case Study," International Journal of Heavy Vehicle Systems, **11**(3,4), pp. 402–432.

[14] Wagner, T. C., 1993, A General Decomposition Methodology for Optimal System Design, Ph.D. thesis, University of Michigan, Ann Arbor, Michigan.

[15] Cooper, A., Georgiopoulos, P., Kim, H. M., and Papalambros, P. Y., 2003, "Analytical Target Setting: An Enterprise Context in Optimal Product Design," *Proceedings of the ASME Design Engineering Technical Conferences*, paper no. DETC 2003/DAC-48734, Chicago, IL.

[16] Wassenaar, H. J., and Chen, W., 2003, "An Approach to Decision-Based Design with Discrete Choice Analysis for Demand Modeling," ASME J. Mech. Des., **125**(3), pp. 490–497.

[17] Gu, X., Renaud, J. E., and Batill, S. M., 1998, "An Investigation of Multidisciplinary Design Subject to Uncertainty," *AIAA-1998-4747,* 7th *AIAA/USAF/NASA/ISSMO Multidisciplinary Analysis and Optimization Symposium*, St. Louise, Missouri, 1998, pp. 309–319.

[18] Chen, W., and Lewis, K., 1999, "Robust Design Approach for Achieving Flexibility in Multidisciplinary Design," AIAA J., **37**(8), pp. 982–989.

[19] Du, X., and Chen, W., 2002, "Efficient Uncertainty Analysis Methods for Multidisciplinary Robust Design," AIAA J., **40**(3), pp. 545–552.

[20] Batill, S., Renaud, J. E., and Gu, X., 2000, "Modeling and Simulation Uncertainty in Multidisciplinary Design Optimization," 8th AIAA/NASA/USAF/ISSMO Multidisciplinary Analysis and Optimization Conference and Exhibit, paper no. AIAA 2000-4803, Long Beach, California.

[21] Padmanabhan, D., and Batill, S. M., 2000, "An Iterative Concurrent Subspace Robust Design Framework," *Proceedings of* 8th *AIAA/NASA/USAF/ISSMO Multidisciplinary Analysis and Optimization Conference and Exhibit*, paper no. AIAA-20002-0449, Long Beach, California.

[22] Gurnani, A. P., and Lewis, K., 2004, "An Approach to Robust Decision Making in Multidisciplinary Selection Problems under Uncertainty," *Proceedings of* 10th *AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference*, August 30–September 1, 2004, Albany, New York.

[23] Sues, R., Cesare, M., Pageau, S., and Wu, J., 2000, "Reliability Based MDO for Aerospace Systems," 8th Symposium on Multidisciplinary Analysis and Optimization, AIAA 2000-4804, Long Beach, California.

[24] Chiralaksanakul, A., and Mahadevan, S., 2004, "Multidisciplinary Design Optimization Under Uncertainty," 10th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, paper no. AIAA-2004-4311, Albany, NY.

[25] Kokkolaras, M., Mourelatos, Z. P., and Papalambros, P. Y., 2006, "Design Optimization of Hierarchically Decomposed Multilevel System under Uncertainty," ASME J. Mech. Des., **128**(2), pp. 503–508. An earlier version was presented at the 2004 ASME Design Engineering Technical Conferences, paper no. DETC2004/DAC-57357.

[26] Phadke, M. S., 1989, *Quality Engineering Using Robust Design*, Prentice Hall, Englewood Cliffs, New Jersey.

[27] Kim, H., Kumar, D., Chen, W., and Papalambros, P. Y., 2004, "Target Feasibility Achievement in Enterprise-Driven Hierarchical Multidisciplinary Design," 10th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, paper no. AIAA-2004-4546, Albany, NY.

[28] Allison, J., 2004, Complex System Optimization: A Comparison of Analytical Target Cascading, Collaborative Optimization, and Other Formulations, MS thesis, University of Michigan, Ann Arbor, Michigan.

[29] Michalek, J. J., and Papalambros, P. Y., 2005, "Technical Brief: Weights, Norms, and Notation in Analytical Target Cascading," ASME J. Mech. Des., **127**(3), pp. 499–501.

[30] Ang, A.H.-S., and Tang, W. H., 1975, *Probability Concepts in Engineering Planning and Design, Volume I–Basic Principles*, Wiley, New York.

[31] Du, X., and Chen, W., 2000, "Towards a Better Understanding of Modeling Feasibility Robustness in Engineering," ASME J. Mech. Des., **122**(4), pp. 385–394.

[32] Kalsi, M., Hacker, K., and Kemper, L., 2001, "A Comprehensive Robust Design Approach for Decision Trade-Offs in Complex Systems Design," ASME J. Mech. Des., **123**, pp. 1–10.

[33] McAllister, C. D., and Simpson, T. W., 2003, "Multidisciplinary Robust Design Optimization of an Internal Combustion Engine," ASME J. Mech. Des., **125**, pp. 124–130.

[34] Du, X., Sudjianto, A., and Chen, W., 2004, "An Integrated Framework for Optimization Using Inverse Reliability Strategy," ASME J. Mech. Des., **126**(4), pp. 562–570.

[35] Wassenaar, H. J., Chen, W., Cheng, J., and Sudjianto, A., 2003, "Enhancing Discrete Choice Demand Modeling for Decision-Based Design," ASME J. Mech. Des., **127**(4), pp. 514–523.

[36] Wassenaar, H. J., Chen, W., Cheng, J., and Sudjianto, A., 2004, "An Integrated Latent Variable Choice Modeling Approach to Enhancing Product Demand Modeling," Proceedings of the ASME Design Engineering Technical Conference, paper no. DETC2004-57487, Salt Lake City, UT.

[37] Michelena, N., Park, H., and Papalambros, P., 2002, "Convergence Properties of Analytical Target Cascading," *Proceedings of the 9th AIAA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, paper no. AIAA-2002-5506, Atlanta, GA; Also in 2003, AIAA J., **41**(5), pp. 897–905.

[38] Haftka, R. T., and Watson, L. T., 2005, "Multidisciplinary Design Optimization with Quasiseparable Subsystems," Opt. Eng., **6**, pp. 9–20.

[39] Beightler, C. S., and Phillips, D. T., 1976, *Applied Geometric Programming*, Wiley, New York.

[40] Tamhane, A. C., and Dunlop, D. D., 2000, *Statistics and Data Analysis from Elementary to Intermediate*, Prentice Hall, New Jersey.

[41] Tosserams, S., Etman, L. F. P., Papalambros, P. Y., and Rooda, J. E., 2005, "An Augmented Lagrangian Relaxation for Analytical Target Cascading using the Alternating Directions Method of Multipliers," *Proceedings of the 6th World Conference on Structural and Multidisciplinary Optimization*, Rio de Janeiro, Brazil.

[42] Kim, H. M., Chen, W., and Wiecek, M., 2005, "Lagrangian Coordination for Enhancing the Convergence of Analytical Target Cascading," *Proceedings of the IFORS*, Honolulu, Hawaii, USA.

[43] Du, X., and Chen, W., 2004, "Sequential Optimization and Reliability Assessment Method for Efficient Probabilistic Design," ASME J. Mech. Des., **126**(2), pp. 225–233.