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ORIGINAL PAPER

Strategic behavior in power markets under uncertainty

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Abstract We consider a setting of a two settlement power market where firms compete in the forward market and an uncertain real-time market. A recourse-based framework is proposed where firms make simultaneous bids in the forward market and take recourse in the real-time market contingent on the realization of uncertainty. The market participants include both generation firms as well as the independent system operator (ISO), the latter of which is assumed to maximize wheeling revenue. The resulting stochastic game-theoretic problem is seen to be a Nash game with coupled strategy sets, often referred to as a generalized Nash game. In general, the primal variational conditions of such problems are given by quasi-variational inequality. Yet, the associated complementarity problem in a primal-dual space admits a monotonicity property that allows us to derive an appropriate existence statement. Computation of equilibria is complicated by the challenge arising from the size of the sample-space. We present a distributed iterative regularization technique that is shown to scale well with the size of the sample-space. Finally, the paper concludes with the application of this model on an electrical network and provides insights on market design and operation.

1 Introduction

With increasing concerns of pollution and environmental impacts from fossil fuels, attention has shifted towards renewable sources of energy. Particularly, as the pen-

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etration of windpower is growing, this raises questions on the associated impact on system reliability, capacity shortfall and consumer welfare in power markets. With growing uncertainty in the generation mix, strategic behavior in power markets can no longer be addressed from a deterministic standpoint and networked stochastic counterparts need to be analyzed. However such models necessarily lead to large-scale problems that are less easy to both analyze and solve.

Game-theoretic formulations have so far proved to be extraordinarily useful tools for analyzing strategic behavior in power markets. Game theory [10, 24] has its roots in the work by von Neumann and Morgenstern [29] while the Nash equilibrium solution concept was forwarded by Nash in 1950 [23]. Pricing of power is a consequence of a series of clearings or settlements. A single settlement structure refers to one where firms bid in the real time market at an endogeneously defined price [11, 12, 22]. A two settlement structure is characterized by firms bidding successively in the forward or day ahead market and the real time or spot market [3, 13, 14, 30]. In this setting, firms are paid at the forward price for their forward bids (promised generation levels) while deviations in the real time market are compensated at the real time price. These games may be analyzed under different levels of rationality. A fully rational model would yield a game where agents compete in the first period market subject to spot market equilibrium. This leads to a challenging class of problems, namely a class of multi-leader multi-follower games. This class of games lead equilibrium program with equilibrium constraints or EPECs. In such games, each agent solves an mathematical program with equilibrium constraints (MPEC) [28, 30, 31]. A bounded rationality framework leads to a game where forward and spot decisions are made simultaneously, leading to variational or complementarity formulations [12, 18, 22], thereby leading to more tractable problems. Lastly models may also be specified by the ISO's objective which may be maximization of social welfare [18, 30, 31] or maximization of wheeling or transmission revenue [11, 22]. A more expansive description of energy models can be found in [2, 15, 26, 27].

The present work extends the deterministic single-settlement framework, proposed in [12, 22], to a boundedly rational two-settlement stochastic regime. In this paper we assume that the ISO maximizes wheeling revenue and we allow for a particular generator to sell power at multiple nodes. This model leads to a Nash game in which agents have coupled constraints which are not common or shared [9]. This comes under a broad and challenging realm of problems, referred to as generalized Nash games [6, 7]. Notably, when these coupled constraints are "shared" in nature, then a solution to an appropriately defined variational inequality gives equilibria of the original shared-constraint game. Here, the equilibrium conditions in the primal space are given by a less tractable quasi-variational inequality. Instead, we opt for analyzing equilibria in the large space of primal and dual variables via a complementarity approach. Surprisingly, under relatively mild conditions, this complementarity problem can be characterized as monotone and is seen to admit solutions. Furthermore, the monotonicity property allows for the construction of distributed regularization techniques. The key contributions of this paper can be viewed as twofold

1. We present a model for capturing strategic behavior in power markets in an uncertain and boundedly rational setting. The model leads to a Nash game with coupled strategy sets that is shown to lead to a complementarity problem that is monotone,

under some conditions. Existence of an equilibrium can be concluded by some additional analysis.

2. Direct solutions of the complementarity problem get increasingly difficult as the sample space grows. Naturally, we consider the development of a decomposition approach that aims to alleviate the challenge arising from problem size. This is achieved through a distributed iterative regularization gradient-based technique. Through numerical tests, this scheme is seen to display the required convergence properties and is observed to scale well with the size of the sample space.

The remainder of the paper is organized into five sections. Section 2 introduces the stochastic two-settlement electricity market model with market clearing conditions. In Sect. 3, we analyze the properties of equilibria arising in such games by examining the properties of the complementarity formulation. A distributed scheme for computing equilibria for this class of problems is derived in Sect. 4. In Sect. 5, we obtain insights through a two-settlement networked electricity market model. We conclude in Sect. 6.

2 Model

A variety of settings have dealt with game-theoretic power market problems. A common deterministic setting is one where firms compete in just a spot market or a single settlement market. However, most power markets are characterized by a multi-settlement framework that naturally requires the incorporation of uncertainty. A two-settlement variant is one where firms bid quantities in the forward market and allow for deviations from these bids in the real time market. Two settlement models differ with regard to the assumptions on rationality. For instance, under complete rationality, firms compete in the forward market, subject to spot market equilibrium. Effectively the problem solved is an equilibrium program with equilibrium constraints (EPEC), where each firm solves a mathematical program with equilibrium constraints (MPEC) [21]. Under the setting of bounded rationality, firms are assumed to simultaneously take decisions in the spot and forward markets. This leads to variational and complementarity formulations.

In this paper, we consider a bounded rationality framework under a Nash-Cournot setting and extend the realm of the single-settlement model in [12, 22] to a two settlement setting. We consider a network where nodes and transmission lines are denoted by $i \in \mathcal{N}$ and $l \in \mathcal{L}$, respectively. (See Table 1 for a list of variables and parameters.) Firm $j \in \mathcal{J}$ bids x_{ij} at node i in the forward market and is paid at a common forward price p_i^0. The spot sales and generation are denoted by s_{ij}^ω and y_{ij}^ω respectively and the associated nodal spot price may be denoted by p_i^ω. Positive and negative deviations are settled at the respective spot prices p_i^ω. This immediately raises a question of arbitrage. A no-arbitrage assumption specifies that the forward price equals expected spot price or

$$p_i^0 = \mathbb{E} p_i^\omega.$$

In practice, arbitrage opportunities exist in power markets and the forward market is cleared independently. In [16], Kamat and Oren refer to this setting as a market-clearing model and allow for an independent prescription of forward and spot prices,

Table 1 Variables and parameters

x_{ij}	Forward sales decision from firm j at node i
s_{ij}^{ω}	Spot sales decision from firm j at node i during scenario ω
$u_{ij}^{\omega}, v_{ij}^{\omega}$	Positive and negative deviations respectively at scenario ω from firm j at node i
$y_{ij}^{\omega}, cap_{ij}^{\omega}$	Total spot generation decisions and capacity at scenario ω for firm j at node i
r_i^{ω}	Import/export at scenario ω at node i
Ω, n, ρ^{ω}	Sample-space, cardinality of sample-space and probability of scenario ω
p_i^{ω}	Price at scenario ω at node i
$c_{ij}^{\omega}, d_{ij}^{\omega}$	Coefficient of linear and quadratic terms in the cost function at scenario ω for firm j at node i
f_p, f_n	Penalty functions for positive and negative deviations
N	Total number of nodes in the network
a_i^0, b_i^0	Intercept and slope of price-function at node i in the forward market
$a_i^{\omega}, b_i^{\omega}$	Intercept and slope of price-function at node i at scenario ω
J	Total number of firms
$Q_{l,i}$	Power flowing across line l due to unit injection/withdrawal of power at node i
K_l^{ω}	Transmission capacity of line l at scenario ω
$\mathcal{N}_j, \mathcal{N}_j^c$	Set of all generating nodes and non-generating nodes for firm j respectively
$\mathcal{J}_i$	Set of all generating firms at node i
$\mathcal{L}, \mathcal{N}$	Set of all transmission lines and set of all nodes respectively
G, G^c	Set of all generating nodes and load nodes respectively
$\mathcal{J}$	Set of all generating firms

defined as

$$p_i^0 \triangleq a_i^0 - b_i^0 \sum_{j \in \mathcal{J}} x_{ij}, \quad \text{and} \quad p_i^{\omega} \triangleq a_i^{\omega} - b_i^{\omega} \sum_{j \in \mathcal{J}} s_{ij}^{\omega}, \tag{1}$$

where a_i^{ω}, a_i^0 and b_i^{ω}, b_i^0 denote the respective intercepts and slopes. To reduce the incentive for purely financial participation and reduce significant changes between forward and spot participation, most markets introduce an additional layer of deviation penalties. This model allows for convex differentiable penalty functions for positive and negative deviations. In addition, each firm incurs a convex generation cost at a generation facility. Firms are permitted to transfer power from the actual generation facilities to other nodes and are charged a transmission fee set by the ISO and given by w_i^{ω}. Imports and exports are charged a fee levied by the ISO. Giving allowance to the fact that multiple firms operate at several nodes in the network, the revenue of agent j may be written as

$$\pi_j \triangleq \sum_{i \in \mathcal{N}} \left(\pi_{ij}^0 + \mathbb{E}\pi_{ij}^{\omega} \right),$$

$$\text{where } \pi_{ij}^0 \triangleq \left(\left(a_i^0 - b_i^0 \sum_{j \in \mathcal{J}} x_{ij} \right) - \mathbb{E}\left(\left(a_i^{\omega} - b_i^{\omega} \sum_{j \in \mathcal{F}} s_{ij}^{\omega} \right) \right) \right) x_{ij}$$

and

$$\pi_{ij}^{\omega} \triangleq \underbrace{\left(a_i^{\omega} - b_i^{\omega} \sum_{j \in \mathcal{J}} s_{ij}^{\omega}\right) s_{ij}^{\omega}}_{\text{Spot revenue}} - \underbrace{C_{ij}^{\omega}(y_{ij}^{\omega})}_{\text{Costs}} - \underbrace{(f_{ij,p}^{\omega}(u_{ij}^{\omega}) + f_{ij,n}^{\omega}(v_{ij}^{\omega}))}_{\text{Deviation penalties}} - \underbrace{w_i^{\omega}(s_{ij}^{\omega} - y_{ij}^{\omega})}_{\text{Wheeling costs}}.$$

The generation costs C_{ij}^{ω} are, in general, assumed to be quadratic, and unless stated otherwise, are defined as

$$C_{ij}^{\omega}(y_{ij}^{\omega}) \triangleq \frac{1}{2} d_{ij}^{\omega}(y_{ij}^{\omega})^2 + c_{ij}^{\omega} y_{ij}^{\omega}. \tag{2}$$

The generation levels of every firm are bounded by their capacities while the total system-wide sales across firms is equal to the total quantity of generation for every firm. The forward and spot sales are related through

$$s_{ij}^{\omega} = x_{ij} + u_{ij}^{\omega} - v_{ij}^{\omega}, \quad \forall i \in \mathcal{N}, \forall j \in \mathcal{J}, \forall \omega \in \Omega.$$

Then, agent j's problem may be compactly represented as follows:

$$\begin{array}{lll} \mathrm{Ag}^j(z_{-j}) & \text{maximize} & \sum_{i \in \mathcal{N}} \left(\mathbb{E}(\pi_{ij}^{\omega}) + \pi_{ij}^{0} \right) \\ & \text{subject to} & \left\{ \begin{array}{ll} y_{ij}^{\omega} \le cap_{ij}^{\omega} & (\alpha_{ij}^{\omega}) \\ s_{ij}^{\omega} - x_{ij} - u_{ij}^{\omega} + v_{ij}^{\omega} \le 0 & (\beta_{ij}^{\omega}) \\ -s_{ij}^{\omega} + x_{ij} + u_{ij}^{\omega} - v_{ij}^{\omega} \le 0 & (\gamma_{ij}^{\omega}) \\ \sum_{i \in \mathcal{N}} y_{ij}^{\omega} - \sum_{i \in \mathcal{N}} s_{ij}^{\omega} \le 0 & (\delta_j^{\omega}) \\ \sum_{i \in \mathcal{N}} s_{ij}^{\omega} - \sum_{i \in \mathcal{N}} y_{ij}^{\omega} \le 0 & (\phi_j^{\omega}) \\ x_{ij}, s_{ij}^{\omega}, y_{ij}^{\omega}, u_{ij}^{\omega}, v_{ij}^{\omega} \ge 0 & \end{array} \right\}, \\ & & \forall i \in \mathcal{N},\ \forall l \in \mathcal{L},\ \forall \omega \in \Omega \end{array}$$

where $z_{-j} \triangleq (z_i)_{i \neq j}$ denotes the tuple of adversarial decisions. Note that the set of equality constraints with respect to s, u, v and x are written as two sets of inequality constraints. Similarly the equality constraints with regard to s and y are written as two sets of inequality constraints. This allows for the formulation of a pure complementarity problem. Note that $\alpha, \beta, \gamma, \delta$ and ϕ represent Lagrange multipliers corresponding to the appropriate constraints.

In a majority of power markets, there exists an ISO that manages the dispatch, pricing and other market level tasks. In several settings, the ISO maximizes social welfare [30, 31]. In this regime, we allow for the ISO to maximize transmission or wheeling charges [12, 22]. Its worth remarking as to why this objective would be an appropriate one for an ISO, in contrast with a social welfare maximizing metric. It has been noted in [22] that such a model is "equivalent to a competitive market for transmission capacity." Naturally, this model prescribes the ISO has a profit-maximizing agent responsible for allocating transmission services, a clear departure from our earlier model.

Power distribution factors, or more specifically the Injection Shift Factors (ISF), may be used to quantify the power flowing across lines in a network. Let Q denote the power distribution factor matrix. Then the power flowing in transmission line l, due to unit injection or withdrawal of power at node i may be denoted by $Q_{l,i}$. The power distribution factor (ISF) is independent of uncertainty and is purely dependent on the network and the choice of the slack node. The details regarding the computation is presented in [20].[1] The ISO's problem may hence be defined as follows:

$$\begin{aligned} \mathrm{Ag}^{J+1}(z_{-(J+1)}) \quad & \text{maximize} \sum_{i\in\mathcal{N}} \mathbb{E}(w_i^{\omega} r_i^{\omega}) \\ & \text{subject to} \quad r_i^{\omega} = \textstyle\sum_{j\in\mathcal{J}} (s_{ij}^{\omega} - y_{ij}^{\omega}) \\ & \qquad\qquad \textstyle\sum_{i\in\mathcal{N}} Q_{l,i} r_i^{\omega} \le K_l^{\omega} \qquad (\mu_l^{\omega}) \\ & \qquad\qquad -\textstyle\sum_{i\in\mathcal{N}} Q_{l,i} r_i^{\omega} \le K_l^{\omega} \quad (\eta_l^{\omega}). \end{aligned}$$

Note that r_i refers to the inflow or outflow corresponding to node i. The strategy set of the ISO consists o the decision variables of the generating firms. However the firms' constraints are independent of the ISO's decisions. This leads to a generalized Nash game with coupled constraints. In general, the equilibria of such games [6, 9] leads to a quasi-variational inequality [8]. Formally the game may be defined as follows:

Definition 1 (Generalized Nash game ($\mathcal{G}$)) The generalized Nash game is given by a collection of $J+1$ agents (J generation firms and the ISO) and denoted by $\mathcal{G}$. Furthermore, an equilibrium of this game is given by a tuple $\{z_1^*, \ldots, z_{J+1}^*\}$ where z_j^* solves the problem $\mathrm{Ag}^j(z_{-j}^*)$ for all $j = 1, \ldots, J+1$ (note that $k = J+1$ regarding the ISO) or

$$z_j^* \in \mathrm{SOL}(Ag^j(z_{-j}^*)), \quad \text{for } j = 1, \ldots, J+1.$$

The analysis of these problems is the focus of the next section.

3 Analysis

In this section, we analyze the generalized Nash game presented in the earlier section. The following assumptions are employed through the remainder of this paper.

[1] A slack node may be one, where injection or withdrawal of power is assumed to have no impact on any line in the network.

Assumption 1

(A1) The cost of generation C_{ij}^{ω} is an increasing convex function of y_{ij}^{ω} for all $i \in \mathcal{N}$, $j \in \mathcal{J}$ and for all $\omega \in \Omega$.
(A2) The nodal forward and spot-market prices are defined by affine price functions (1) for all $i \in \mathcal{N}$ and for all $\omega \in \Omega$.
(A3) The slopes, $b_i^0, b_i^{\omega} \geq 0, \forall i \in \mathcal{N}, \forall \omega \in \Omega$. Moreover, the forward slopes for all $i \in \mathcal{N}$ are defined such that, $b_i^0 \geq \frac{1}{4}\mathbb{E}b_i^{\omega}$.
(A4) The deviation penalty functions $f_{ij,p}$ and $f_{ij,n}$ are convex increasing functions of u_{ij}^{ω} and v_{ij}^{ω} for all $i \in \mathcal{N}$, $j \in \mathcal{J}$ and $\omega \in \Omega$.

Unfortunately, the presence of independent forward prices leads to a bilinear term in the agent objectives. However, under a suitable assumption, the convexity of this problem can be claimed.

Lemma 1 *Suppose assumptions* (A1)–(A4) *hold. Then the objective functions of the generation firms are convex.*

Proof With convex generation costs and linear wheeling charges it suffices to prove the convexity of the expectation term of every agent's objective, given by $\eta_{ij}(x_{ij}, y_{ij}; y_{i,-j})$, defined as

$$\eta_{ij}(x_{ij}, s_{ij}; x_{i,-j}, s_{i,-j}) = -\left(a_i^0 - b_i^0 \sum_{j \in \mathcal{J}} x_{ij}\right) x_{ij} - \sum_{\omega \in \Omega} \rho^{\omega} \left(a_i^{\omega} - b_i^{\omega} \left(\sum_{j \in \mathcal{J}} s_{ij}^{\omega}\right)\right)(s_{ij}^{\omega} - x_{ij}).$$

The gradient and Hessian of this function are given by

$$\nabla \eta_{ij} = \begin{pmatrix} b_i^0 x_{ij} + b_i^0 \sum_{j \in \mathcal{J}} x_{ij} - a_i^0 + \sum_{\omega \in \Omega} \rho^{\omega} a_i^{\omega} - \sum_{\omega \in \Omega} \rho^{\omega} b_i^{\omega} (\sum_{j \in \mathcal{J}} s_{ij}^{\omega}) \\ \rho^{\omega}(-a_i^1 + b_i^1 (s_{ij}^1 + \sum_{j \in \mathcal{J}} s_{ij}^1) - b_i^1 x_{ij}) \\ \vdots \\ \rho^n(-a_i^n + b_i^n (s_{ij}^n + \sum_{j \in \mathcal{J}} s_{ij}^n) - b_i^n x_{ij}) \end{pmatrix},$$

and

$$\nabla^2 \eta_{ij} = \begin{pmatrix} b_i^0 & -\rho^1 b_i^1 & \dots & -\rho^n b_i^n \\ -\rho^1 b_i^1 & 2\rho^1 b_i^1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\rho^n b_i^n & 0 & \dots & 2\rho^n b_i^n \end{pmatrix},$$

respectively.

Let m be an arbitrary nonzero vector. Then by adding and subtracting terms, we have

$$
\begin{aligned}
m^T \nabla^2 \eta_{ij} m &= 2b_i^0 m_1^2 - 2m_1 \sum_{\omega=1}^{n} \rho^\omega b_i^\omega m_{\omega+1} + 2\sum_{\omega=1}^{n} \rho^\omega b_i^\omega m_{\omega+1}^2 \\
&= \left(2b_i^0 - \sum_{\omega=1}^{n} \rho^\omega \frac{b_i^\omega}{2}\right) m_1^2 + \sum_{\omega=1}^{n} \rho^\omega \frac{b_i^\omega}{2} m_1^2 \\
&\quad - 2m_1 \sum_{\omega=1}^{n} \rho^\omega b_i^\omega m_{\omega+1} + 2\sum_{\omega=1}^{n} \rho^\omega b_i^\omega m_{\omega+1}^2 \\
&= \left(2b_i^0 - \sum_{\omega=1}^{n} \rho^\omega \frac{b_i^\omega}{2}\right) m_1^2 + \sum_{\omega=1}^{n} \rho^\omega b_i^\omega \left(\frac{m_1}{\sqrt{2}} - \sqrt{2} m_{\omega+1}\right)^2 .
\end{aligned}
$$

By assumption $\mathbb{E} b_i^\omega \le 4b_i^0$ for all i. This implies that $m^T \nabla^2 \eta_{ij} m \ge 0$ for all nonzero m and $\eta_{ij}(x_{ij}, s_{ij}; s_{i,-j})$ is a convex function in x_{ij} and s_{ij} for all fixed $x_{i,-j}$ and $s_{i,-j}$. The convexity of the agent objectives follow. □

The convexity of the agent problems implies that the aggregated first-order Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient. Furthermore, this aggregated system may be analyzed for purposes of deriving existence statements. We assume that the penalty functions for deviations are quadratic. Then the aggregated KKT conditions are given by the following complementarity problem for all i, j, ω:

$$
\begin{aligned}
& 0 \le x_{ij} \perp \sum_{j \in \mathcal{J}} b_i^0 x_{ij} + b_i^0 x_{ij} - a_i^0 + \sum_{\omega \in \Omega} \rho^\omega \left(a_i^\omega - b_i^\omega \sum_{j \in \mathcal{J}} s_{ij}^\omega \right) \\
& \qquad - \sum_{\omega \in \Omega} \beta_{ij}^\omega + \sum_{\omega \in \Omega} \gamma_{ij}^\omega \ge 0 \\
& 0 \le s_{ij}^\omega \perp \rho^\omega \left(b_i^\omega \sum_{j \in \mathcal{J}} s_{ij}^\omega + b_i^\omega s_{ij}^\omega - b_i^\omega x_{ij} - a_i^\omega \right) + \sum_{l \in \mathcal{L}} Q_{l,i} (\mu_l^\omega - \eta_l^\omega) - \delta_j^\omega + \phi_j^\omega \\
& \qquad + \beta_{ij}^\omega - \gamma_{ij}^\omega \ge 0 \\
& 0 \le y_{ij}^\omega \perp \rho^\omega (d_{ij}^\omega y_{ij}^\omega + e_{ij}^\omega) - \sum_{l \in \mathcal{L}} Q_{l,i} (\mu_l^\omega - \eta_l^\omega) + \alpha_{ij}^\omega + \delta_j^\omega - \phi_j^\omega \ge 0 \\
& 0 \le u_{ij}^\omega \perp \rho^\omega \left(e_{ij}^\omega u_{ij}^\omega + h_{ij}^\omega \right) - \beta_{ij}^\omega + \gamma_{ij}^\omega \ge 0 \\
& 0 \le v_{ij}^\omega \perp \rho^\omega \left(o_{ij}^\omega v_{ij}^\omega + t_{ij}^\omega \right) + \beta_{ij}^\omega - \gamma_{ij}^\omega \ge 0 \\
& 0 \le \alpha_{ij}^\omega \perp cap_{ij}^\omega - y_{ij}^\omega \ge 0 \\
& 0 \le \beta_{ij}^\omega \perp x_{ij} + u_{ij}^\omega - v_{ij}^\omega - s_{ij}^\omega \ge 0
\end{aligned}
$$

$$0 \le \gamma_{ij}^{\omega} \perp -x_{ij} - u_{ij}^{\omega} + v_{ij}^{\omega} + s_{ij}^{\omega} \ge 0$$

$$0 \le \delta_j^{\omega} \perp \sum_{i \in \mathcal{N}} s_{ij}^{\omega} - \sum_{i \in \mathcal{N}} y_{ij}^{\omega} \ge 0$$

$$0 \le \phi_j^{\omega} \perp \sum_{i \in \mathcal{N}} y_{ij}^{\omega} - \sum_{i \in \mathcal{N}} s_{ij}^{\omega} \ge 0$$

$$0 \le \mu_l^{\omega} \perp K_l^{\omega} - \sum_{i \in \mathcal{N}} Q_{l,i} \sum_{j \in \mathcal{J}} (s_{ij}^{\omega} - y_{ij}^{\omega}) \ge 0$$

$$0 \le \eta_l^{\omega} \perp K_l^{\omega} + \sum_{i \in \mathcal{N}} Q_{l,i} \sum_{j \in \mathcal{J}} (s_{ij}^{\omega} - y_{ij}^{\omega}) \ge 0.$$

Note that $0 \le x \perp y \ge 0$ implies that $x, y \ge 0$ and $x^T y = 0$ and w_i^{ω} has been eliminated from the ISO's equilibrium conditions. Furthermore, the notation for x, y, u and v follows that of s.

$$z = \begin{pmatrix} p \\ d \end{pmatrix}, \qquad p = \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix}, \qquad p_i = \begin{pmatrix} x_i \\ s_i \\ y_i \\ u_i \\ v_i \end{pmatrix}, \qquad s_i = \begin{pmatrix} s_i^1 \\ \vdots \\ s_i^n \end{pmatrix},$$

$$s_i^{\omega} = \begin{pmatrix} s_{i1}^{\omega} \\ \vdots \\ s_{iJ}^{\omega} \end{pmatrix}, \qquad d = \begin{pmatrix} d_1 \\ \vdots \\ d_N \\ \delta \\ \phi \\ \mu \\ \eta \end{pmatrix}, \qquad d_i = \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} \alpha_i^1 \\ \vdots \\ \alpha_i^n \end{pmatrix},$$

$$\alpha_i^{\omega} = \begin{pmatrix} \alpha_{i1}^{\omega} \\ \vdots \\ \alpha_{iJ}^{\omega} \end{pmatrix}, \qquad \delta = \begin{pmatrix} \delta^1 \\ \vdots \\ \delta^n \end{pmatrix}, \qquad \delta^{\omega} = \begin{pmatrix} \delta_1^{\omega} \\ \vdots \\ \delta_J^{\omega} \end{pmatrix},$$

$$\mu = \begin{pmatrix} \mu^1 \\ \vdots \\ \mu^n \end{pmatrix}, \qquad \mu^{\omega} = \begin{pmatrix} \mu_1^{\omega} \\ \vdots \\ \mu_L^{\omega} \end{pmatrix}.$$

The resulting problem can be cast as a linear complementarity problem, denoted by LCP(q, M). Such a problem [5] requires a vector z satisfying $0 \le z \perp Mz + q \ge 0$ where M is defined as

$$M \triangleq \begin{pmatrix} M_p & -M_d^T \\ M_d & 0 \end{pmatrix}.$$

The matrices M_p and M_d are given by

$$M_p = \begin{pmatrix} M_{p,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & M_{p,N} \end{pmatrix} \quad \text{and} \quad M_{p,i} = \begin{pmatrix} \bar{M}_{p,i} & 0 \\ 0 & T_i \end{pmatrix}, \quad \text{where}$$

$$\bar{M}_{p,i} = \begin{pmatrix} N_i^0 & P_i^1 & \dots & P_i^n \\ R_i^1 & N_i^1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_i^n & 0 & \dots & N_i^n \end{pmatrix} \quad \text{and} \quad T_i = \begin{pmatrix} T_{y,i} & 0 & 0 \\ 0 & T_{u,i} & 0 \\ 0 & 0 & T_{v,i} \end{pmatrix}.$$

It can be seen that the matrix M_p is an asymmetric coefficient matrix corresponding to the primal constraints and primal variables. The matrix M_d refers to the coefficient matrix corresponding to the dual constraints and primal variables. Furthermore $T_{u,i}$, $T_{v,i}$ and $T_{y,i}$ represent diagonal matrices. If I and e refer to an identity matrix and a column vector of ones, respectively, then the components of $\bar{M}_{p,i}$ and T_i can be defined as follows.

$$\begin{aligned}
&T_{u,i} = \operatorname{diag}\left(T_{u,i}^1 \dots T_{u,i}^n\right), \qquad T_{v,i} = \operatorname{diag}\left(T_{v,i}^1 \dots T_{v,i}^n\right),\\
&T_{y,i} = \operatorname{diag}\left(T_{y,i}^1 \dots T_{y,i}^n\right), \qquad T_{u,i}^\omega = \operatorname{diag}\left(\rho^\omega e_{i1}^\omega \dots \rho^\omega e_{iJ}^\omega\right),\\
&T_{v,i}^\omega = \operatorname{diag}\left(\rho^\omega o_{i1}^\omega \dots \rho^\omega o_{iJ}^\omega\right), \qquad T_{y,i} = \operatorname{diag}\left(\rho^\omega d_{i1}^\omega \dots \rho^\omega d_{iJ}^\omega\right),\\
&P_i^\omega = -\rho^\omega b_i^\omega ee^T, \qquad N_i^\omega = \rho^\omega b_i^\omega (I + ee^T), \qquad N_i^0 = b_i^0 (I + ee^T),\\
&R_i^\omega = -\rho^\omega b_i^\omega I.
\end{aligned}$$

Finally, the matrix M_d and its associated components are defined as follows.

$$M_d = \begin{pmatrix} D_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & D_N \\ E_1 & \dots & E_N \\ -E_1 & \dots & -E_N \\ F_1 & \dots & F_N \\ -F_1 & \dots & -F_N \end{pmatrix}, \qquad D_i = \begin{pmatrix} 0 & 0 & -I & 0 & 0 \\ I & -I & 0 & I & -I \\ -I & I & 0 & -I & I \end{pmatrix},$$

$$E_i = \begin{pmatrix} 0 & I & -I & 0 & 0 \end{pmatrix},$$

$$F_i = \begin{pmatrix} 0 & -K_i & K_i & 0 & 0 \end{pmatrix}, \qquad K_i = \begin{pmatrix} Q_i & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & Q_i \end{pmatrix},$$

$$Q_i = \begin{pmatrix} Q_{1,i} & \dots & Q_{1,i} \\ \vdots & \ddots & \vdots \\ Q_{L,i} & \dots & Q_{L,i} \end{pmatrix}.$$

Using these definitions, we show that M is a positive semidefinite matrix.

Lemma 2 *Suppose* (A1)–(A4) *hold. Then M is a positive semidefinite matrix.*

Proof With assumptions on convexity of costs and deviation penalties, it suffices to show that $\bar{M}_{p,i}$ is positive semidefinite. Let m be an arbitrary column vector. It follows that

$$\begin{aligned} m^T \bar{M}_{p,i} m = &\left(b_i^0 - \sum_{\omega=1}^{n} \frac{\rho^\omega b_i^\omega}{4}\right) \sum_{k=1}^{g} m_k^2 + \left(b_i^0 - \sum_{\omega=1}^{n} \frac{\rho^\omega b_i^\omega}{4}\right) \left(\sum_{k=1}^{g} m_k\right)^2 \\ &- \sum_{\omega=1}^{n} \rho^\omega b_i^\omega \sum_{k=1}^{g} m_k m_{\omega g+k} \\ &+ \sum_{\omega=1}^{n} \frac{\rho^\omega b_i^\omega}{4} \sum_{k=1}^{g} m_k^2 + \sum_{\omega=1}^{n} \frac{\rho^\omega b_i^\omega}{4} \left(\sum_{k=1}^{g} m_k\right)^2 \\ &- \sum_{\omega=1}^{n} \rho^\omega b_i^\omega \sum_{k=1}^{g} m_k \sum_{k=1}^{g} m_{\omega g+k} \\ &+ \sum_{\omega=1}^{n} \rho^\omega b_i^\omega \left(\sum_{k=1}^{g} m_{\omega g+k}^2 + \left(\sum_{k=1}^{g} m_{\omega g+k}\right)^2\right). \end{aligned}$$

Combining terms and completing the squares we get the following expression:

$$\begin{aligned} m^T \bar{M}_{p,i} m = &\left(b_i^0 - \sum_{\omega=1}^{n} \frac{\rho^\omega b_i^\omega}{4}\right) \sum_{k=1}^{g} m_k^2 + \left(b_i^0 - \sum_{\omega=1}^{n} \frac{\rho^\omega b_i^\omega}{4}\right) \left(\sum_{k=1}^{g} m_k\right)^2 \\ &+ \sum_{\omega=1}^{n} \left(\rho^\omega b_i^\omega \sum_{k=1}^{g} \left(\frac{m_k}{2} - m_{\omega g+k}\right)^2\right) \\ &+ \sum_{\omega=1}^{n} \rho^\omega b_i^\omega \left(\sum_{k=1}^{g} m_{\omega g+k} - \sum_{k=1}^{g} \frac{m_k}{2}\right)^2 \geq 0. \end{aligned}$$

This completes the proof. □

The monotonicity of LCP(q, M) follows. However, existence of a solution to this LCP can be concluded by noting that $M \in \mathbf{Q}_0$, a class of matrices for which feasibility of the LCP is sufficient for solvability [5, Theorem 3.1.2].

Theorem 1 (Existence of Equilibria) *Consider the game $\mathcal{G}$ and suppose* (A1)–(A4) *hold. Then $\mathcal{G}$ admits an equilibrium.*

Proof It suffices to show that there exists a nonnegative vector $z^{\text{ref}} \in \mathbb{R}_+^{\bar{N}}$ such that $(Mz^{\text{ref}} + q) \geq 0$. Before proceeding, we recall that z and q are given by

$$z = \begin{pmatrix} x \\ s \\ y \\ u \\ v \\ \alpha \\ \beta \\ \gamma \\ \delta \\ \phi \\ \mu \\ \eta \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} q_x \\ q_s \\ q_y \\ q_u \\ q_v \\ q_\alpha \\ q_\beta \\ q_\gamma \\ q_\delta \\ q_\phi \\ q_\mu \\ q_\eta \end{pmatrix},$$

respectively and z_β denotes the set of components of z corresponding to a subvector β. For instance $z_x = x$ and $z_\beta = \beta$. Next we define $s^{\text{ref}}, y^{\text{ref}}, u^{\text{ref}}, \beta^{\text{ref}}, \gamma^{\text{ref}}, \delta^{\text{ref}}, \mu^{\text{ref}}, \eta^{\text{ref}} \triangleq 0$. Then, it is seen that $q_p \geq 0$ and $(Mz^{\text{ref}} + q)_p \geq 0$ for $p = u, v, \alpha, \delta, \phi, \mu, \eta$. Furthermore, it is seen that q_x is negative. Thus setting

$$x_{ij}^{\text{ref}} \triangleq \frac{a_i^0}{b_i^0} \quad \Longrightarrow \quad (Mz^{\text{ref}} + q)_x > 0.$$

However a positive value of x makes $(Mz^{\text{ref}} + q)_\gamma$ negative. Setting

$$v_{ij,\omega}^{\text{ref}} = x_{ij}^{\text{ref}} \quad \Longrightarrow \quad (Mz^{\text{ref}} + q)_\beta, (Mz^{\text{ref}} + q)_\gamma = 0.$$

A positive value of x also turns $(Mz^{\text{ref}} + q)_s$ negative. Setting

$$\phi_{j,\omega}^{\text{ref}} \triangleq \max_{i \in \mathcal{N}} \rho^\omega a_i^\omega + \max_{i \in \mathcal{N}} \rho^\omega b_i^\omega \frac{a_i^0}{b_i^0} + \Delta \quad \Longrightarrow \quad (Mz^{\text{ref}} + q)_s > 0.$$

A positive value of ϕ turns $(Mz^{\text{ref}} + q)_y$ negative. Setting

$$\alpha_{ij,\omega}^{\text{ref}} \triangleq \max_{i \in \mathcal{N}} \rho^\omega a_i^\omega + \max_{i \in \mathcal{N}} \rho^\omega b_i^\omega \frac{a_i^0}{b_i^0} + 2\Delta \quad \Longrightarrow \quad (Mz^{\text{ref}} + q)_y > 0,$$

where $\Delta > 0$. It is seen that z^{ref} satisfies both $(Mz^{\text{ref}} + q)_p \geq 0$ and $z_p^{\text{ref}} \geq 0$ for all $p = x, s, y, u, v, \alpha, \beta, \gamma, \delta, \phi, \eta, \mu$ and is feasible with respect to LCP(q, M). This completes the proof. $\square$

We conclude this section by observing that the monotonicity of the mapping allows one to claim that a regularized game admits a unique equilibrium in a primal-dual space.

Proposition 1 (Uniqueness of Equilibria) *Consider the game $\mathcal{G}$ and suppose* (A1)–(A4) *hold. Suppose $\mathcal{G}_\epsilon$ pertains to game whose associated complementarity problem is denoted by LCP$(q, M + \epsilon I)$. Then $\mathcal{G}_\epsilon$ admits a unique equilibrium for every $\epsilon > 0$.*

Proof This follows from noting that $M + \epsilon I$ is positive definite and the required existence and uniqueness result follows immediately from Theorem 3.1.6 [5]. □

4 Distributed regularization schemes for computing equilibria

A crucial part of this paper pertains to computing equilibria in these regimes. In this section, we examine how one may develop schemes for computing solutions to complementarity problems, particularly in stochastic regimes. A host of techniques exist for computing solutions to monotone linear complementarity problems (cf. [5, 8]). Unfortunately, these methods rely on the solution of systems that can grow to be massive as Ω grows in cardinality. Therefore a direct application of these techniques proves inadvisable and alternate schemes that rely on decomposition must be introduced. Past work by Shanbhag et al. on monotone stochastic LCPs [28] introduced a matrix-splitting framework for computing solutions. More recently, a distributed method for monotone variational problems [18] has been suggested. This scheme relies on projections and uses cutting-plane techniques to alleviate the growth in complexity. Both schemes display a linear growth in computational effort as the cardinality of Ω grows.

In general, the application of projection-based techniques with constant steplengths require that the mappings be strongly monotone. However, in our regime, the mapping is merely monotone. This challenge is overcome in [18] through regularization. Specifically, we construct primal-dual and dual constant steplength methods for regularized problems and provide error bounds to relate the regularized solution to its original counterpart. However, the goal in that paper was to develop constant steplength schemes for solving regularized problems where the regularization parameter was fixed a priori.

In this paper, we consider an alternate avenue lies in driving the regularization parameter to zero or by using a proximal-point method. Both techniques are investigated in detail in [17, 33] and are applied here. Here, we pursue a different avenue in which we update the regularization parameter at every step. Motivated by the above issues, we present iterative regularization counterparts of the Tikhonov regularization method and the proximal-point method. These schemes are eponymously termed as the iterative Tikhonov regularization scheme (ITR) and the iterative proximal point (IPP) scheme.

Note that LCP(M, q) is equivalent to VI$(\mathbb{R}^{\bar{N}}_+, Mz + q)$ [8]. Furthermore, z^* is a solution to the VI$(\mathbb{R}^{\bar{N}}_+, Mz + q)$ if and only if

$$z^* = \Pi_{\mathbb{R}^+_{\bar{N}}}(z^* - \gamma F(z^*)), \tag{3}$$

where $F(z) = Mz + q$ and $\gamma > 0$. A regularization to a monotone mapping yields a strongly monotone mapping. The standard Tikhonov scheme rests on solving a sequence of such well-posed regularized problems where the regularized map is given by $F^k(z) \triangleq F(z) + \epsilon^k z$. Accordingly, the iterates are defined by

$$z^k = \Pi_{\mathbb{R}^+_{\bar{N}}} \left(z^k - \gamma (F(z^k) + \epsilon^k z^k) \right) \quad k \geq 1.$$

Furthermore, if $\lim_{k\to\infty} \epsilon_k = 0$, then we have that $\lim_{k\to\infty} z^k = z^*$, as shown in [8, Chap. 12]. An alternative lies in the proximal-point framework in which the regularized subproblems have a different structure. Specifically, strong monotonicity in this framework is maintained by using a map $F^k(z) \triangleq F(z) + \epsilon^k (z - z^k)$ where z^k is referred to as the centering parameter. Accordingly, the regularized fixed-point problem is given by

$$z^k = \Pi_{\mathbb{R}_{\bar{N}}{}^+} \left(z^k - \gamma F(z^k) + \theta (z^k - z^{k-1}) \right),$$

and the iterate z^k is a solution of $\mathrm{VI}(K, F^k))$. Convergence theory for both regularization schemes, when applied to monotone variational inequalities, can be found in [1, 8].

In this paper, we consider iterative regularization counterparts of Tikhonov and proximal-point schemes. Here, the regularization parameter (in the case of Tikhonov) and the centering parameter (in the case of proximal-point) are updated at every iterate, rather than when a fixed point is available. Such iterative regularization techniques have a long history in optimization (cf. [25]) yet have been less used in the solution of variational inequalities, barring work by Konnov [19]. In recent work [17, 33], we have examined the convergence of properties of iterative Tikhonov and proximal-point schemes in the context of monotone Nash games. In the next two subsections, we briefly state and describe each of these schemes. Finally, a discussion of how these schemes scale with $|\Omega|$ is provided.

4.1 Iterative Tikhonov regularization scheme

The original Tikhonov scheme is a two-timescale scheme that rests on solving a strongly monotone variational inequality, at every step. In general, this may be a computationally challenging requirement. We alleviate this problem by requiring that a

Algorithm 1: Iterative Tikhonov Regularization Scheme

```
0  initialization k = 0;
   choose constants ψ, Δ, ε_0, γ_0 > 0, z^0 ≥ 0 and α ∈ (0.5, 1), β ∈ (0, 0.5);
   while ‖F̄^nat(z^k, F^k)‖ > Δ do
1      z^{k+1} = Π_{R_+^N̄} (z^k − γ_k(F(z^k) + ε_k z^k));
2      Update regularization ε_{k+1} := ε_0 / (k+1)^β;
3      Update step size γ_{k+1} := γ_0 / (k+1)^α;
4      Compute ‖F̄^nat(z^k, F^k)‖ := ‖z^k − Π_Z (z^k − ψ(F(z^k)))‖;
5      k := k + 1;
   end
```

single projection step be taken along with an update of the regularization parameter. Formally, the algorithm is stated as follows: Note that the value of ψ is taken to be 1 in general and Δ denotes the stopping criterion. If $\Delta = 0$, it implies that the solution is a fixed point of the problem. The following theorem from [32] and [17] specifies the requirements on the step size and the regularization parameter to obtain a convergent solution.

Theorem 2 (Convergence of ITR scheme) *Consider the VI*$(\mathbf{Z}, \mathbf{F})$. *Let the step sizes be defined such that* $\sum_{k=1}^{\infty} \gamma_k = \infty$ *and* $\sum_{k=1}^{\infty} \gamma_k^2 < \infty$. *In addition, let* $\sum_{k=1}^{\infty} \epsilon_k \gamma_k = \infty$. *Further let* $\epsilon^k \to 0$ *and*

$$\lim_{k \to \infty} \frac{\gamma_k}{\epsilon_k} = 0.$$

Let F *be monotone on* Z *that is convex. In addition, let* F *be Lipschitz continuous with constant* L. *Then,* z^k *converges to* z^*, *where* z^* *is given by the fixed point relation in* (3).

Proof See appendix. □

Its not immediately obvious that there are indeed acceptable steplength and regularization sequences. The next lemma from [33] demonstrates this is indeed the case.

Lemma 3 (Acceptable steplength sequences) *Consider the choice of step sizes and regularizations of the form* $\gamma_k = \frac{\gamma_0}{k^\alpha}$ *and* $\epsilon^k = \frac{\epsilon_0}{k^\beta}$. *Let* $0.5 < \alpha < 1$ *and* $0 < \beta < 0.5$. *Then the parameters satisfy the requirements stated in Theorem 2.*

4.2 Iterative proximal scheme

Next, we present a single timescale version of the proximal point method. Just as in the ITR scheme, each iterate is given by a projection step, defined formally as

$$z^{k+1} = \Pi_{\mathbb{R}^+_{\bar{N}}} \left(z^k - \gamma_k (F(z^k) + \theta(z^k - z^{k-1})) \right), \quad k \geq 0.$$

Algorithm 2: Iterative Proximal Point (IPP) Algorithm

initialization $k = 0$;
choose constants $\psi, \Delta, \theta, \gamma_0 > 0$, $z^0 \geq 0$ and $\alpha \in (0.5, 1)$;
while $\|\bar{F}^{nat}(z^k, F^k)\| > \Delta$ **do**
  $z^{k+1} = \Pi_Z \left(z^k - \gamma_k (F(z^k) + \theta(z^k - z^{k-1})) \right)$;
  Update step size $\gamma_{k+1} := \frac{\gamma_0}{(k+1)^\alpha}$;
  Compute $\|\bar{F}^{nat}(z^k, F^k)\| = \left\| z^k - \Pi_Z \left(z^k - \psi (F(z^k) + \theta(z^k - z^{k-1})) \right) \right\|$;
  $k := k + 1$;
end

The convergence of the scheme for diminishing steplengths has been proved in [17] and the formal result is stated as follows:

Theorem 3 (Convergence of IPP Scheme) *Let the mapping F be strictly monotone on Z that is convex. In addition, let F be Lipschitz continuous with constant L and let Z be closed and compact. Suppose γ_k satisfies*

$$\sum_{k=1}^{\infty} \gamma_k = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \gamma_k^2 < \infty.$$

Let $\theta > 0$ be a fixed parameter. Then $z^k \to z^*$, where z^* refers to a solution to the $VI(Z, F)$.

Proof See appendix. □

Note that in a complementarity regime where F is merely monotone and Z is merely closed, the IPP scheme need not converge. In practice, however, we observe that it performs well on the class of problems of interest.

Remark Projection over sets that may be characterized by a large number of constraints, as is common in stochastic regimes, has been dealt with in the past through a variety of decomposition schemes such as cutting plane methods [4]. However, an advantage of the current setting is that the projection problems are over the nonnegative orthant and are therefore fairly cheap and parallelizable operations. However, it should be noted that with this reduction in complexity comes with a growth in size, since we now consider a larger problem in the primal-dual space.

4.3 Numerical experiments

This section analyzes the performance of the proposed algorithms. The scalability of the iterative regularization schemes was tested by applying it to large scale stochastic problems where $|\Omega|$ may be large. The iterative Tikhonov and the iterative proximal schemes were compared for different test cases confined to the network with 12 nodes and 13 transmission lines. The grid details are shown in Table 2. Node 12 was chosen to be the slack node. Four generators were assumed to compete in the market, the details of which are mentioned in Table 3. The spot-price intercepts were taken to be 700 across all nodes and scenarios while the forward and spot price slopes were taken to be drawn from normal distribution of given by $N(1, 0.02)$ across all nodes and all scenarios. Linear and quadratic deviation penalties were taken to be $N(8, 0)$ and $N(8, 0)$ respectively for all generators at all nodes and scenarios. Finally, the schemes were implemented on Matlab 7.0 on a Linux OS machine with a clockspeed of 2.39 GHZ and a memory of 16 GB.

Scalability The initial step lengths and regularizations were taken to be, $\gamma^0 = 0.15$ and $\epsilon^0 = 0.25$ respectively for all the runs. The order of decrease of step lengths was taken to be $\beta = 0.5001$. The order of decrease of the regularization parameter was

Table 2 Network details

Line	Imp. (Ohm)	Cap. (MW)
1–2	11000	400
2–3	8500	480
3–4	8000	440
4–5	7000	440
1–3	9000	480
1–6	10000	520
6–7	6000	360
7–8	8000	400
8–9	6500	340
9–10	9500	380
4–10	8500	420
9–11	8000	460
10–12	7000	500

Table 3 Generator details

Generator type	Capacity	Linear costs	Quadratic costs
1	$N(2000, 10)$	$N(2, 0)$	$N(8, 0)$
2	$N(2000, 10)$	$N(2, 0)$	$N(8, 0)$
3	$N(650, 270)$	$N(2, 0)$	$N(8, 0)$
4	$N(730, 320)$	$N(2, 0)$	$N(8, 0)$

taken to be $\alpha = 0.498$. For the first set, we fixed the number of firms to be three (Firms 1, 2 and 3). For two different values of the forward intercepts (a^0), we varied the number of scenarios from 5 to 60 in steps of 5. It is to be noted that the scheme is distributed and computation can be done in parallel. However, we proceed to show that even serial times scale well with the size of the problem. The stopping tolerance Δ was taken to be proportional to the problem size.

$$\Delta = \|F^{nat}(z)\| = \|z - \max(z - F(z), 0)\| \leq \frac{|\Omega|}{10}.$$

Table 4 reports the corresponding serial computation time and final regularization values for all instances and scenarios.

Comparison between schemes A four firm problem, under the same setting was taken as a case study to compare the two schemes. The schemes were tested by varying the number of scenarios from 5 to 60 in steps of 5 for three different instances. The initial step size was the same ($\gamma_0 = 0.15$) for both the schemes while θ was taken to be 10 for the IPP scheme. The stopping criterion was taken to be the same in both the cases. The results are reported in Table 5. The IPP scheme shows a better performance terms of the number of iterations.

Table 4 Scalability with $|\Omega|$

	No. of scenarios	Variables	Serial time (s)	Iterations
$a^0 = 900$	5	1456	123.48	596251
	10	2876	139.24	272007
	15	4296	181.96	178963
	20	5716	195.36	135193
	25	7136	248.77	124511
	30	8556	379.70	162553
	35	9976	576.26	210889
	40	11396	875.62	278733
	45	12816	1296.48	366574
	50	14236	1814.03	443753
	55	15656	2331.13	528315
	60	17076	3069.52	607701
$a^0 = 950$	5	1456	139.10	665701
	10	2876	138.98	300640
	15	4296	200.14	197092
	20	5716	214.89	148469
	25	7136	259.14	130812
	30	8556	383.25	164232
	35	9976	584.05	212735
	40	11396	873.52	279423
	45	12816	1307.09	369709
	50	14236	1833.24	449557
	55	15656	2361.93	538197
	60	17076	3122.51	617349

Table 5 Comparison: ITR and IPP schemes

	No. of scenarios	Iterations	
		Iterative Tikhonov	Iterative Proximal
$a^0 = 900$	10	511402	46273
	15	472182	59406
	20	1029679	66484
$a^0 = 950$	10	532134	49293
	15	481456	61836
	20	1032890	71708

5 Insights

Though the above stated schemes perform well with regard to large problems, second-order solvers prove to be more efficient for smaller scale problems. In this section,

Table 6 Forward participation and premium—no deviation penalties

Intercepts	Node 1		Node 2		Node 3	
	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$
50	0.00	−183.99	0.00	−183.99	0.00	−183.99
100	0.00	−133.99	0.00	−133.99	0.00	−133.99
150	0.00	−83.99	0.00	−83.99	0.00	−83.99
200	0.00	−33.99	0.00	−33.99	0.00	−33.99
250	12.95	3.21	13.35	3.23	12.77	3.20
300	53.37	13.23	55.03	13.30	52.65	13.19
350	93.80	23.25	96.72	23.37	92.52	23.19
400	134.22	33.27	138.40	33.45	132.40	33.18
450	174.65	43.28	180.09	43.52	172.27	43.18
500	215.08	53.30	221.77	53.60	212.15	53.17

we use KNITRO (v 5.0) as our solver to solve the exact LCP(q, M). This case study uses the same generator details mentioned in Table 3. The number of scenarios was taken to be twenty ($n = 20$). However, in this case, the linear and quadratic costs were taken to be $N(2, 0)$ and $N(0.2, 0)$ respectively. Unless stated, the linear and quadratic deviation penalties (positive and negative) were taken to be $N(2, 0)$ and $N(0.2, 0)$ respectively. We focus on two major questions pertaining to two settlement markets. First, to what extent do forward prices impact participation? Second, what impact does increasing wind power penetration have on forward participation?

5.1 Forward commitments

We define a term called nodal premium, given by the difference between the forward price and the expected spot price at that node. The deviation penalties were set to be zero and the spot intercepts were fixed to be 700. The forward intercepts across all nodes were varied from 50 to 500 in steps of 50. Table 6 shows the variation of forward bids with increasing forward intercepts across nodes 1, 2 and 3. Firms do not bid in the forward market, till a particular level is reached where they find a positive premium. The same behavior is seen across the other nodes. However with sufficiently high deviation penalties, the behavior is not the same. Firms bid in the forward market even when they do not find a premium, in order to decrease losses due to positive deviation. Results with the above assumed deviation penalties are reported in Table 7.

5.2 Wind power penetration

Here, we fix the capacity levels of generators 1, 2 and 3 while the capacity of the wind (fourth) generator is varied from $\mathcal{N}$ (30, 10) to $\mathcal{N}$ (300, 90). It is seen that the forward commitments of the firms tend to increase (Fig. 1). Increased volatility and reduced spot prices may be attributed to be reasons for this behavior. In addition, it is seen that the mean premium tends to increase.

Table 7 Forward participation and premium—quadratic deviation penalties

Intercepts	Node 1		Node 2		Node 3	
	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$
50	0.00	−183.97	0.00	−183.93	0.00	−184.01
100	0.00	−133.97	0.00	−133.93	0.00	−134.01
150	0.00	−83.97	0.00	−83.93	0.00	−84.01
200	0.00	−33.97	0.00	−33.93	0.00	−34.01
250	31.36	−14.98	32.32	−14.85	30.95	−15.06
300	70.23	−3.41	72.39	−3.18	69.29	−3.53
350	109.10	8.16	112.46	8.49	107.63	8.00
400	147.97	19.73	152.53	20.15	145.98	19.53
450	186.84	31.30	192.60	31.82	184.32	31.06
500	225.70	42.88	232.67	43.49	222.66	42.59

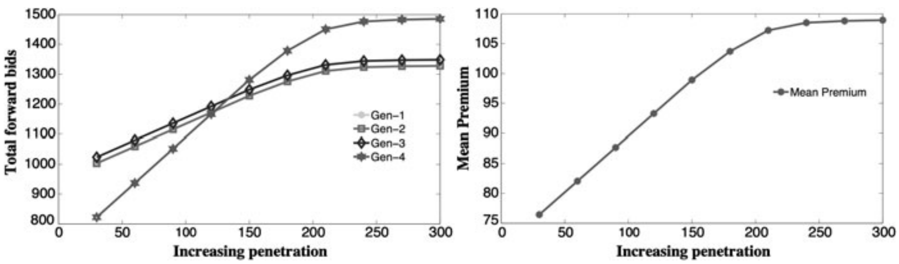

Fig. 1 Penetration of wind power

5.3 Policy insights

Several insights may be drawn from the standpoint of market design and operations, particularly in the face of integrating volatile renewables. Most markets introduce a penalty that is charged on deviations in the real-time market from forward positions. If these penalties are set at zero, then there is no forward participation unless the expected spot prices exceed forward prices. If, however, the deviation penalties are raised to a sufficiently high level, then participants bid even when forward prices are less than expected spot prices; this ensures that participants can better recuperate penalties. It remains a goal of future work to study this relationship closer, primarily because deviation penalties represent a potentially useful tool in managing risk associated with shortfalls in real-time markets. A second observation pertains to the increase in forward participation when the volatility and penetration increases. With the growth in wind capacity, this appears to be a natural consequence. In accordance with a growth in the mean capacity, we notice that forward participation also increases. This is partly a consequence of the need to sell cheap power without paying significant deviation costs.

6 Summary

A two settlement structure with uncertainty is considered where agents compete in the forward and spot markets. Under an assumption of bounded rationality, agents make bids in the forward market and provide a simultaneous recourse-based bid in the real-time market. Furthermore, the ISO is assumed to maximize wheeling revenue. The resulting Nash game is seen to have coupled strategy sets and belongs to a class of generalized Nash games. The agent objective functions are shown to be convex and the resulting complementarity formulation proves to be more tractable. In fact, the mapping of the LCP proves to be monotone, a property that proves useful in developing an existence result.

However the absence of strong monotonicity rules out the avenue of traditional projection algorithms. Motivated by this shortcoming, this paper discusses two different convergent schemes namely the iterative Tikhonov regularization and the iterative proximal point algorithms. It is seen from the numerical results that the algorithms scale very well with the problem size and comparison tests show that the IPP algorithm is more effective.

Lastly, some insights are obtained from the above model by applying it to a 12-node network. It is also observed that in the absence of exogenous deviation penalties, firms do not bid in the forward market unless they see an incentive. The same is not seen to be the case in a setting with deviation penalties. Moreover with increasing wind penetration it is seen that the market becomes more volatile and the firms bid more in the forward market. It is also seen that with increasing volatility due to wind assets, the risk premium tends to increase.

Appendix

The following Lemmas from [25] are employed in developing our convergence theory.

Lemma 4 *Let* $u_{k+1} \le q_k u_k + \alpha_k$, $0 \le q_k < 1$, $\alpha_k \ge 0$ *and*

$$\sum_{k=0}^{\infty}(1-q_k) = \infty, \qquad \frac{\alpha_k}{1-q_k} \to 0, \quad k \to \infty.$$

Then $\lim_{k\to\infty} u_k \le 0$ *and if* $u_k > 0$, *then* $\lim_{k\to\infty} u_k = 0$.

Lemma 5 *Let* $u_{k+1} \le (1+v_k)u_k + p_k$, $u_k, v_k, p_k \ge 0$ *and*

$$\sum_{k=0}^{\infty} v_k < \infty, \qquad \sum_{k=0}^{\infty} p_k < \infty, \quad k \to \infty.$$

Then $\lim_{k\to\infty} u_k = \bar{u} \ge 0$.

Our proof of convergence relies on relating the iterates of the proposed ITR scheme to that of the original Tikhonov scheme. The following Lemma reproduced from [32] provides a bound between consecutive iteratives of the standard Tikhonov scheme.

Lemma 6 *Let the mapping* ∇F *be monotone and suppose* $\mathrm{SOL}(Z, F)$ *be nonempty and bounded. Consider the standard exact Tikhonov scheme defined by iterates* $\{y^k\}$. *If* $M := \|z^*\|_2$, *then*

$$\|y^k - y^{k-1}\| \leq \frac{M(\epsilon^{k-1} - \epsilon^k)}{\epsilon^k}.$$

Proof Omitted (see [32]). □

Proof for Theorem 2

For the sake of completeness, the following proof is reproduced from [32] and [17]. The regularization parameter $\{\epsilon^k\}$ and the steplength sequence $\{\gamma_k\}$ are assumed to satisfy the following assumption.

Assumption 2 (A5) The mapping $F(\mathbf{x})$ is Lipschitz continuous with constant L. The regularization parameter ϵ^k and steplength γ_k satisfy $\sum_{k=1}^{\infty} \gamma^k \epsilon^k = \infty$, $\epsilon^{k+1} \leq \epsilon^k, \forall k$, $\gamma_{k+1} \leq \gamma_k \forall k$, $\lim_{k\to\infty} \gamma_k/\epsilon^k = 0$, $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$, $\sum_{k=0}^{\infty} (\gamma_k \epsilon^k)^2 < \infty$ and

$$\lim_{k\to\infty} \frac{\epsilon^{k-1} - \epsilon^k}{\gamma_k (\epsilon^k)^2} = 0. \tag{4}$$

Let y^k denote the standard Tikhonov iterate. By the triangle inequality, $\|z^{k+1} - z^*\|$ can be bounded by terms 1 and 2:

$$\|z^{k+1} - z^*\| \leq \underbrace{\|z^{k+1} - y^k\|}_{\text{term 1}} + \underbrace{\|y^k - z^*\|}_{\text{term 2}}.$$

Of these, term 2 converges to zero from the convergence statement of Tikhonov regularization methods. It suffices to show that term 1 converges to zero as $k \to \infty$ which follows as shown next. By using the non-expansivity of the Euclidean projector, $\|z^{k+1} - y^k\|^2$ is given by

$$\begin{aligned}
\|z^{k+1} - y^k\|^2 &= \left\| \Pi_Z\left(z^k - \gamma_k(F(z^k) + \epsilon^k z^k)\right) - \Pi_Z\left(y^k - \gamma_k(F(y^k) + \epsilon^k y^k)\right) \right\|^2 \\
&\leq \left\| \left(z^k - \gamma_k(F(z^k) + \epsilon^k z^k)\right) - \left(y^k - \gamma_k(F(y^k) + \epsilon^k y^k)\right) \right\|^2.
\end{aligned}$$

This expression can be further simplified as

$$\begin{aligned}
&\|(1 - \gamma_k \epsilon^k)(z^k - y^k) - \gamma_k(F(z^k) - F(y^k))\|^2 \\
&= (1 - \gamma_k \epsilon^k)^2 \|z^k - y^k\|^2 + \gamma_k^2 \|F(z^k) - F(y^k)\|^2 \\
&\quad - 2\gamma_k(1 - \gamma_k \epsilon_k)(z^k - y^k)^T(F(z^k) - F(y^k)) \\
&\quad \leq (1 - 2\gamma_k \epsilon_k + \gamma_k^2(L^2 + \epsilon_k^2))\|z^k - y^k\|^2,
\end{aligned}$$

where the last inequality follows from $\gamma_k \epsilon_k \le 1$ and the monotonicity of $F(x)$ over Z. If Lemma 4 can indeed be invoked then it follows that $\|z^{k+1} - y^k\| \to 0$ as $k \to \infty$. The remainder of the proof shows that the conditions for employing Lemma 4 do hold. It can be seen that

$$\begin{aligned}\|z^{k+1} - y^k\| \le q_k \|z^k - y^k\| &\le q_k \|z^k - y^{k-1}\| + q_k \|y^k - y^{k-1}\| \\ &\le q_k \|z^k - y^{k-1}\| + q_k M \frac{(\epsilon^{k-1} - \epsilon^k)}{\epsilon^k},\end{aligned}$$

where the second inequality is a consequence of the triangle inequality and the third inequality follows from Lemma 6. Suppose $q_k := \sqrt{(1 - 2\gamma_k \epsilon_k + \gamma_k^2 (L^2 + \epsilon_k^2)}$. Invoking Lemma 4 requires showing that

$$\sum_{k=0}^{\infty} (1 - q_k) = \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{q_k}{1 - q_k} M \frac{(\epsilon^{k-1} - \epsilon^k)}{\epsilon^k} = 0.$$

It is easily seen that

$$\begin{aligned}\sum_{k=0}^{\infty} (1 - q_k) = \sum_{k=0}^{\infty} \frac{1 - q_k^2}{1 + q_k} &= \sum_{k=0}^{\infty} \left(\frac{2\gamma_k \epsilon_k - \gamma_k^2 (L^2 + \epsilon_k^2)}{1 + q_k} \right) \\ &> \sum_{k=0}^{\infty} (2\gamma_k \epsilon_k - \gamma_k^2 (L^2 + \epsilon_k^2)) = \infty,\end{aligned}$$

where the inequality follows from $q_k < 1$ and the final equality follows from $\sum_{k=0}^{\infty} \gamma_k \epsilon^k = \infty$ and the square summability of $\gamma_k \epsilon_k$ and γ^k. The second requirement follows by observing that

$$\begin{aligned}\frac{q_k}{1 - q_k} M \frac{(\epsilon^{k-1} - \epsilon^k)}{\epsilon^k} &= \frac{q_k (1 + q_k)}{1 - q_k^2} M \frac{(\epsilon^{k-1} - \epsilon^k)}{\epsilon^k} \\ &= \frac{q_k (1 + q_k)}{2\gamma_k \epsilon_k - \gamma_k^2 (L^2 + \epsilon_k^2)} M \frac{(\epsilon^{k-1} - \epsilon^k)}{\epsilon^k} \\ &= \underbrace{\frac{q_k (1 + q_k)}{2 - \frac{\gamma_k}{\epsilon^k} (L^2 + \epsilon_k^2)}}_{\text{Term 1}} M \underbrace{\frac{(\epsilon^{k-1} - \epsilon^k)}{\gamma_k (\epsilon^k)^2}}_{\text{Term 2}}.\end{aligned}$$

Since $\gamma_k, \epsilon^k \to 0$, it follows that $q_k \to 1$. Since $\gamma_k / \epsilon^k \to 0$, Term 1 tends to 1 as $k \to \infty$. By assumption (A5), Term 2 tends to zero as $k \to \infty$. The proof for the existence of such a sequence is not straightforward as stated earlier and is proved in [32]. □

Proof for Theorem 3

For the sake of completeness, the following proof is reproduced from [17]. We begin by expanding $\|z^{k+1} - z^*\|$ and by using the non-expansivity property of projection.

$$\begin{aligned}\|z^{k+1} - z^*\|^2 &= \|\Pi_Z(z^k - \gamma_k(F(z^k) + \theta(z^k - z^{k-1}))) - \Pi_Z(z^* - \gamma_k F(z^*))\|^2 \\ &\le \|(z^k - z^*) - \gamma_k(F(z^k) - F(z^*)) - \gamma_k\theta(z^k - z^{k-1})\|^2.\end{aligned}$$

Expanding the right hand side,

$$\begin{aligned}\|z^{k+1} - z^*\|^2 \le \|z^k - z^*\|^2 &+ (\gamma_k L)^2\|z^k - z^*\|^2 + (\gamma_k\theta)^2\|z^k - z^{k-1}\|^2 \\ &- 2\gamma_k(z^k - z^*)^T(F(z^k) - F(z^*)) \\ &- 2\gamma_k\theta(z^k - z^{k-1})^T\left(z^k - z^* - \gamma_k(F(z^k) - F(z^*))\right).\end{aligned}$$

Using Lipschitz and monotonicity properties of $F(x)$, we have

$$\begin{aligned}\|z^{k+1} - z^*\|^2 \le (1 + \gamma_k^2 L^2)\|z^k - z^*\|^2 &+ (\gamma_k\theta)^2\|z^k - z^{k-1}\|^2 \\ &\underbrace{-2\gamma_k\theta(z^k - z^{k-1})^T\left((z^k - z^*) - \gamma_k(F(z^k) - F(z^*))\right)}_{\text{Term 1}}.\end{aligned}$$

Term 1 can bounded from above by the use of the Cauchy-Schwartz inequality, the boundedness of the iterates, namely $\|z^k - z^*\| \le C$, and the Lipschitz continuity of F, as shown next.

$$\begin{aligned}\|z^{k+1} - z^*\|^2 &\le (1 + \gamma_k^2 L^2)\|z^k - z^*\|^2 + (\gamma_k\theta)^2\|z^k - z^{k-1}\|^2 \\ &\quad + 2\gamma_k\theta\|z^k - z^{k-1}\|\left(\|z^k - z^*\| + \gamma_k\|F(z^k) - F(z^*)\|\right) \\ &\le (1 + \gamma_k^2 L^2)\|z^k - z^*\|^2 + (\gamma_k\theta)^2\|z^k - z^{k-1}\|^2 \\ &\quad + 2\gamma_k\theta C\|(z^k - z^{k-1})\| + 2\gamma_k^2\theta LC\|(z^k - z^{k-1})\|.\end{aligned}$$

Next, we derive a bound on $\|z^k - z^{k-1}\|$ by leveraging the non-expansivity of the Euclidean projector.

$$\begin{aligned}\|z^k - z^{k-1}\| &= \|\Pi_Z(z^{k-1} - \gamma_{k-1}(F(z_{k-1}) + \theta(z^{k-1} - z^{k-2}))) - \Pi_Z(z^{k-1})\| \\ &\le \|(z^{k-1} - \gamma_{k-1}(F(z_{k-1}) + \theta(z^{k-1} - z^{k-2}))) - (z^{k-1})\| \\ &= \| - \gamma_{k-1}(F(z_{k-1}) + \theta(z^{k-1} - z^{k-2}))\|.\end{aligned}$$

It follows from the boundedness of Z and the continuity of $F(z)$, that there exists a $\beta > 0$ such that $\|F(z)\| \le \beta$ for all $z \in Z$, implying that $\|z^k - z^{k-1}\| \le \gamma_{k-1}(\beta + \theta C)$.

The bound on $\|z^k - z^{k-1}\|$ allows us to derive an upper bound $\|z^{k+1} - z^*\|^2$:

$$\begin{aligned}
\|z^{k+1} - z^*\|^2 &\leq (1 + \gamma_k^2 L^2)\|z^k - z^*\|^2 + (\gamma_k \theta)^2 \gamma_{k-1}^2 (\beta + \theta C)^2 \\
&\quad + 2\gamma_k \gamma_{k-1} \theta C (1 + \gamma_k L)(\beta + \theta C) \\
&\leq (1 + \underbrace{\gamma_k^2 L^2}_{\triangleq v_k})\|z^k - z^*\|^2 \\
&\quad + \underbrace{(\gamma_k \theta)^2 \gamma_k^2 (\beta + \theta C)^2 + 2\gamma_k^2 \theta C (1 + \gamma_k L)(\beta + \theta C)}_{\triangleq p_k}.
\end{aligned}$$

The above sequence can be compactly represented as the recursive sequence $u_{k+1} \leq (1 + v_k)u_k + p_k$, where

$$\sum_{k=0}^{\infty} v_k = L^2 \sum_{k=0}^{\infty} \gamma_k^2 < \infty, \qquad \sum_{k=0}^{\infty} p_k < \infty,$$

the latter a consequence of the square summability of γ_k. It follows from Lemma 5 that $u_k \to \bar{u} \geq 0$. It remains to show that $\bar{u} = 0$. Recall that $\|z^{k+1} - z^*\|^2$ is bounded as per the following expression:

$$\|z^{k+1} - z^*\|^2 \leq (1 + v_k)\|z^k - z^*\|^2 + p_k - 2\gamma_k (z^k - z^*)^T (F(z^k) - F(z^*)).$$

Suppose $\bar{u} > 0$. It follows that along every subsequence, we have that $\mu_k = (z_k - z^*)^T (F(z_k) - z^*) \geq \mu' > 0, \forall k$. This is a consequence of the strict monotonicity of F whereby if $(F(z^k) - F(z^*))^T (z^k - z^*) \to 0$ if $z^k \to z^*$. Since $\bar{u} > 0$, it follows that $\|z^k - z^*\|^2 \to \bar{u} > 0$. Then by summing over all k, we obtain

$$\lim_{k \to \infty} \|z^{k+1} - z^*\|^2 \leq \|z^0 - z^*\|^2 + \sum_{k=0}^{\infty} \gamma_k^2 L^2 \|z^k - z^*\|^2 - 2 \sum_{k=0}^{\infty} \gamma_k \mu_k + \sum_{k=0}^{\infty} p_k.$$

Since v_k and p_k are summable and $\mu_k \geq \mu' > 0$ for all k, we have that:

$$\begin{aligned}
\lim_{k \to \infty} \|z^{k+1} - z^*\|^2 &\leq \|z^0 - z^*\|^2 + \sum_{k=0}^{\infty} \gamma_k^2 L^2 \|z^k - z^*\|^2 - 2 \sum_{k=0}^{\infty} \gamma_k \mu_k + \sum_{k=0}^{\infty} p_k \\
&\leq \|z^0 - z^*\|^2 + \sum_{k=0}^{\infty} \gamma_k^2 L^2 \|z^k - z^*\|^2 - 2\mu' \sum_{k=0}^{\infty} \gamma_k \\
&\quad + \sum_{k=0}^{\infty} p_k \leq -\infty,
\end{aligned}$$

where the latter follows from observing that $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$, $\sum_{k=0}^{\infty} \gamma_k = \infty$ and $\|z_k - z^*\| \leq C$. But this is a contradiction, implying that along some subsequence, we have that $\mu_k \to 0$ and $\liminf_{k\to\infty} \|z^k - z^*\|^2 = 0$. But we know that $\{z^k\}$ has a limit point and that the sequence z_k converges. Therefore, we have that $\lim_{k\to\infty} z_k = z^*$. □

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